The alternation number of a knot
(結び目の交代化数について)

理学研究科
数物系専攻

平成２１年度

Tetsuya Abe
(安部 哲哉)

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Contents

Chapter 1

Introduction

A *link* is a disjoint union of circles embedded in S^3, a *knot* is a link with one component and a *tangle* is a 1-manifold properly embedded in a 3-ball. Throughout this paper, all links are oriented. The *alternation number* of a link L, denoted by $\mathrm{alt}(L)$, is the minimal number of crossing changes needed to deform L into an alternating link, which measures the distance from the alternating links and was introduced by A. Kawauchi [29]. Kawauchi gives a lower bound for the alternation number of a link by using a variant of the Nakanishi index, and showed the existence of a hyperbolic link with alternation number n by using the lower bound and the imitation theory for each positive integer n [29].

In Chapter 2, we give a lower bound for the alternation number of a knot. More precisely, we show that the difference of the Rasmussen s-invariant [54] and the negative signature of a knot, denoted by $-\sigma$, gives a lower bound for the alternation number of a knot.

Theorem 2.4. *We have*

$$\left|\frac{s(K) - (-\sigma(K))}{2}\right| \leq \mathrm{alt}(K) \tag{1.1}$$

for any knot K.

We note that the value

$$\left|\frac{s(K) - (-\sigma(K))}{2}\right|$$

is an integer since the values $s(K)$ and $-\sigma(K)$ are even integers.

Theorem 2.4 is based on the two facts. One is that the Rasmussen s-invariant and the negative signature of a knot have the same value on all alternating knots [54]. This implies that a knot K is not alternating if

$$s(K) \neq -\sigma(K).$$

Note that S. Baader [7] also used this fact to construct some non-alternaitng knots. The other is the following [54]:

$$0 \leq s(K_+) - s(K_-) \leq 2,$$

$$0 \leq -\sigma(K_+) - (-\sigma(K_-)) \leq 2,$$

where K_+ and K_- are knots which differ by one crossing change from a positive crossing in K_+ to a negative one in K_-.

In Chapter 3, we study the alternation number of a torus knot. Throughout this paper, we assume p and q are coprime integers such that $2 \leq p < q$. We denote by $T_{p,q}$ the (p, q)-torus knot. Our main result is that we determine the torus knots with alternation number one.

Theorem 3.6. *We obtain the followings.*
(1) $\mathrm{alt}(T_{p,q}) = 0 \Longleftrightarrow p = 2$.
(2) $\mathrm{alt}(T_{p,q}) = 1 \Longleftrightarrow (p,q) = (3,4)$ *or* $(3,5)$.
(3) $\mathrm{alt}(T_{p,q}) \geq 2 \Longleftrightarrow$ *otherwise.*

Furthermore we obtain that many torus knots are indeed "far" from the alternating knots.

Theorem 3.7. *For each positive integer n, there exists a positive integer N such that for all $p > N$, we have $n \leq \mathrm{alt}(T_{p,q})$.*

There is a similar numerical invariant, the *dealternating number*, which is the minimal number of crossing changes needed to change a diagram of a link into an alternating diagram [4] (see also [5], p. 144). The minimum is taken over all possible diagrams of the link. Note that we do not permit using any type of Reidemeister moves in the definition of the dealternating number. We denote the dealternating number of a link L by $\mathrm{dalt}(L)$. Then we have

$$\mathrm{alt}(L) \leq \mathrm{dalt}(L) \tag{1.2}$$

for any link L. We note that the alternation number and the dealternating number are different invariants. Indeed Proposition 2.8 shows there are infinitely many knots K such that

$$\mathrm{alt}(K) < \mathrm{dalt}(K).$$

An *almost alternating* link is a link with dealternating number one. As an application of Theorem 3.6, we obtain

Corollary 3.8 ([1]). *The almost alternating torus knots are just $T_{3,4}$ and $T_{3,5}$.*

This corollary gives an affirmative answer for a conjecture in [4] (see also [5], p. 142). The author has subsequently learned that M. Stošić [58] has a different proof of corollary 3.8.

In Chapter 4, we recall an upper bound for the dealternating number of a positve closed 3-braid (Lemma 4.1). As an application, we determine the dealternating numbers and the alternation numbers of some closed positive 3-braid knots as follows.

Theorem 4.4. *Let β be a 3-braid of the form*

$$\Delta^{2n} \prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i}$$

such that $\widehat{\beta}$ is a knot, $n \geq 0$ and $p_i, q_i \geq 2$ for $i = 1, 2, \ldots, r$. Then we have

$$\mathrm{alt}(\widehat{\beta}) = \mathrm{dalt}(\widehat{\beta}) = n + r - 1.$$

Notations used in the above theorem is explained in Chapter 4.

In Chapter 5, we show that Montesinos links are alternating or almost alternating [4] (see also [2]). In Chapter 6, we study the Turaev genus of a knot, which is related to the alternation number of a knot.

Notation. For a knot K, we denote by $g(K)$ its genus, by $g_*(K)$ its four ball genus. We denote by $\mathrm{Con}(S^3)$ the knot concordance group. For a diagram D, we denote by $c(D)$ is the crossing number of D and by $o(D)$ the number of Seifert circles of D.

Acknowledgments

The author would like to express his sincere gratitudes to Akio Kawauchi and Taizo Kanenobu for helpful advices and comments. He would like to thank the members of Friday Seminar on Knot Theory in Osaka City University, especially Atsushi Ishii, In Dae Jong and Kengo Kishimoto. He also thanks Gyo Taek Jin, Yasutaka Nakanishi, Alexander Stoimenow for helpful advices and comments.

Chapter 2

Lower bound for the alternation number of a knot

In this section, we define the L-property for a $\mathbb{Z}$-valued knot invariant and give a lower bound for the alternation number of a knot.

A $\mathbb{Z}$-valued knot invariant ν has the *L-property* if ν satisfies the following conditions (i) - (iii):

i. the map ν induces a homomorphism from $\mathrm{Con}(S^3)$ to $\mathbb{Z}$,

ii. for any knot K, we have $|\nu(K)| \leq g_*(K)$,

iii. $\nu(T_{2,3}) = 1$.

The following theorem is essentially due to C. Livingston.

Lemma 2.1. *Let ν and ν' be $\mathbb{Z}$-valued knot invariants which have the L-property. If $\nu(L) = \nu'(L)$ for all alternating knots L, then we have*

$$|\nu(K) - \nu'(K)| \leq \mathrm{alt}(K)$$

for any knot K.

Proof. Set $|\nu(K) - \nu'(K)| = n$. The case $n = 0$ is trivial. We consider the case $n > 0$. By the proof of Corollary 3 in [36], we have

$$0 \leq \nu(K_+) - \nu(K_-) \leq 1, \tag{2.1}$$

$$0 \leq \nu'(K_+) - \nu'(K_-) \leq 1. \tag{2.2}$$

Let K_1 be a knot obtained from K by one crossing change. By equalities (2.1) and (2.2), we have

$$\left|(\nu(K) - \nu'(K)) - (\nu(K_1) - \nu'(K_1))\right| \leq 1.$$

By using the triangle inequality, we have

$$\left||\nu(K) - \nu'(K)| - |\nu(K_1) - \nu'(K_1)|\right| \leq 1. \tag{2.3}$$

Let K_{n-1} be a knot obtained from K by $n-1$ crossing changes. By the equality (2.3), we have $|\nu(K_{n-1}) - \nu'(K_{n-1})| \geq 1$. Then K_{n-1} is not alternating because if K_{n-1} is alternating, we have $|\nu(K_{n-1}) - \nu'(K_{n-1})| = 0$ by the hypothesis. This implies that we need, at least, n crossing changes to deform K into an alternating knot. □

2.1 Knot invariants

We recall some knot invariants from the viewpoint of the L-property, and give a proof of Corollary 2.4.

It is well known that the negative half of the signature $-\sigma/2$ has the L-property [28].

Lemma 2.2 ([52], [54]). *The following knot invariants have the L-property.*

i. The half of the Rasmussen s-invariant.

ii. The Ozsváth-Szabó τ-invariant.

It is also well known that the Rasmussen s-invariant of the (p,q)-torus knot is the following:

$$s(T_{p,q}) = (p-1)(q-1). \tag{2.4}$$

The equality (2.4) and the condition (ii) of the L-property for the Rasmussen s-invariant give a simple proof of the Milnor conjecture which states the unknotting number of the (p,q)-torus knot is $(p-1)(q-1)/2$ which was first proved by P. Kronheimer and T. Mrowka via Donaldson theory [32].

The invariants s, σ and τ are closely related:

Lemma 2.3 ([50], [54]). *For all alternating knots K, we have*

$$s(K)/2 = -\sigma(K)/2 = \tau(K).$$

Theorem 2.4. *We have*

$$\left|\frac{s(K) - (-\sigma(K))}{2}\right| \leq \mathrm{alt}(K) \tag{2.5}$$

for any knot K.

Proof. By Lemmas 2.2 and 2.3, the pair $s(K)/2$ and $-\sigma(K)/2$ satisfies the assumption of Lemma 2.1. □

Remark 2.5. The pairs $(s/2, \tau)$ and $(\tau, -\sigma/2)$ also satisfy the assumption of Theorem 2.1.

2.2 Applications

In this section, we study the table of knots up to 11 crossings, and give some applications of Theorem 2.4.

We first study the table of knots up to 11 crossings.

Proposition 2.6. *Let K be a knot up to 11 crossings. Then we have*

$$\mathrm{alt}(K) \leq 1.$$

Proof. All knots K up to 11 crossings are alternating or almost alternating expect 11_{n95} and 11_{n118} (see [5] or [20]). This implies $\mathrm{alt}(K) \leq 1$ for the knots K. Figures 1 and 2 imply that $\mathrm{alt}(11_{n95}) = 1$ and $\mathrm{alt}(11_{n118}) = 1$.

□

There are 22 knots K such that $s(K) \neq -\sigma(K)$ up to 11 crossings [54]. Table 2.1 is the list of the 22 knots. The following is a simple application of Theorem 2.4.

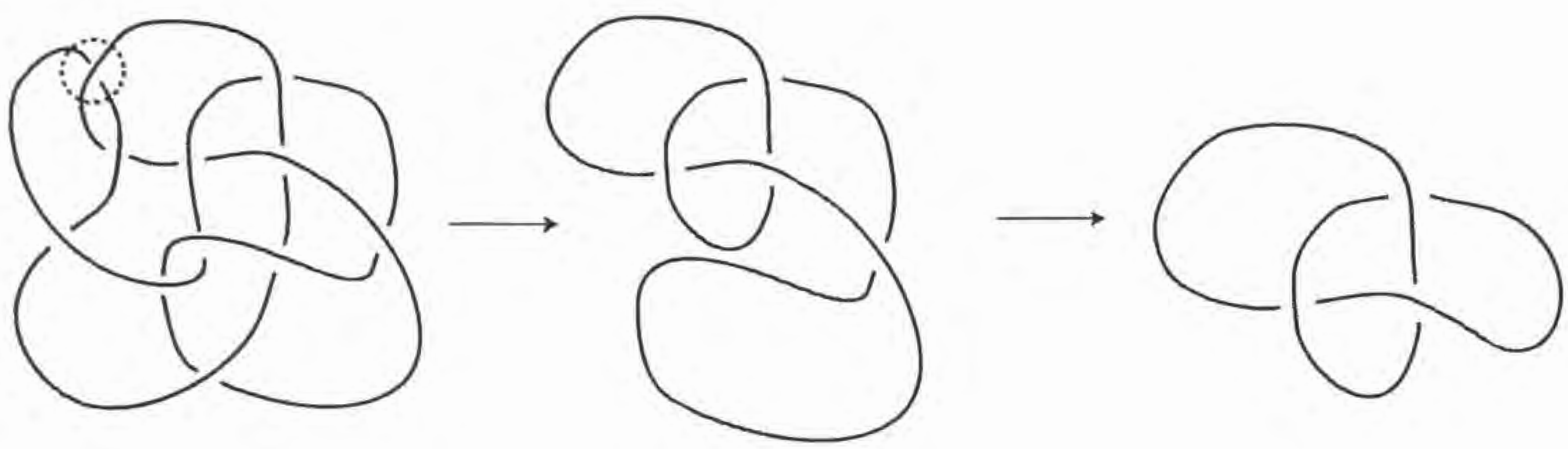

Figure 2.1: $\mathrm{alt}(11_{n95}) = 1$

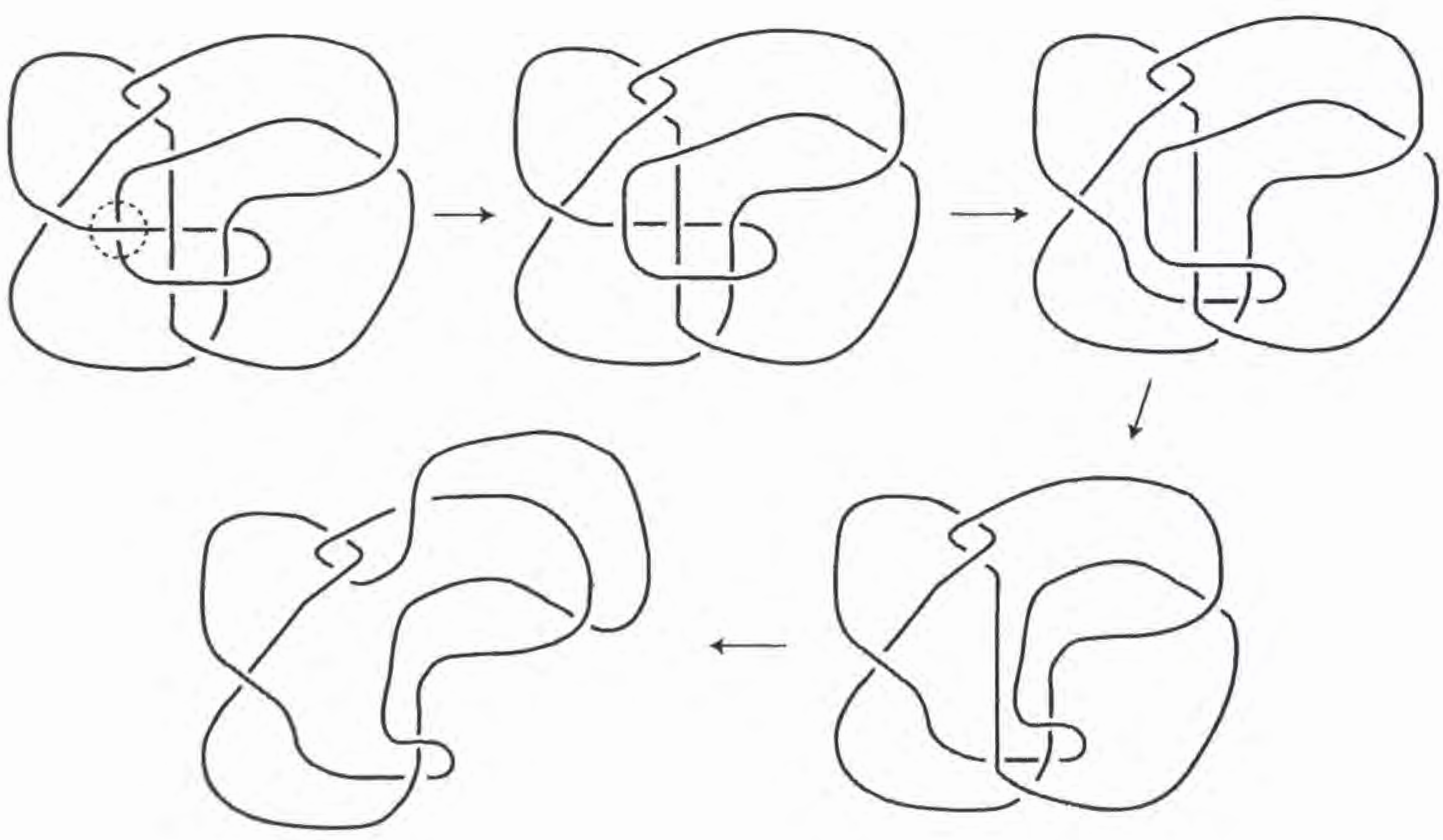

Figure 2.2: $\mathrm{alt}(11_{n118}) = 1$

Proposition 2.7. *We have*

$$\mathrm{alt}(\#^n 9_{42}) = \mathrm{dalt}(\#^n 9_{42}) = n,$$

where $\#^n 9_{42}$ *is the connected sum of* n *copies of* 9_{42}.

Proof. It is easy to see that $\mathrm{dalt}(\#^n 9_{42}) \leq n$ from the diagram of $\#^n 9_{42}$. On the other hand, we have $|s(9_{42}) - (-\sigma(9_{42}))| = 2$. Due to the condition (i) of the L-property for $s/2$ and $-\sigma/2$, we have $|s(\#^n 9_{42}) - (\sigma(-\#^n 9_{42}))| = 2n$. This implies $n \leq \mathrm{alt}(\#^n 9_{42})$ by Corollary 2.4. Therefore we have $n \leq \mathrm{alt}(\#^n 9_{42}) \leq \mathrm{dalt}(\#^n 9_{42}) \leq n$.

□

Let $D_+(L,t)$ be a t-twisted positive Whitehead double of a knot L.

Proposition 2.8. *There are infinitely many knots* K *such that*

$$\mathrm{alt}(K) = 1 \text{ and } \mathrm{dalt}(K) > 1.$$

Proof. Suppose that L is a non-trivial knot. Then the unknotting number of $D_+(L,t)$ is one, and so $D_+(L,t)$ is prime [28]. Since a prime satellite knot is not alternating [40], $D_+(L,t)$ is not alternating. Thus we have $\mathrm{alt}(D_+(L,t)) = 1$. Next, since a prime satellite knot is not almost alternating [4], we have $\mathrm{dalt}(D_+(L,t)) > 1$. If L' and L'' are not equivalent, then $D_+(L',t)$ and $D_+(L'',t)$ are not equivalent [55], and thus the proof is compete. □

Remark 2.9. M. Hedden and P. Ording [23] showed $s(D_+(T_{2,5},5))/2 = 1$ and $\tau(D_+(T_{2,5},5)) = 0$. By this fact and Remark 2.5, we have

$$\mathrm{alt}(\#^n D_+(T_{2,5},5)) = n.$$

K	$s(K)$	$-\sigma(K)$	$\lvert s(K) - (-\sigma(K))\rvert/2$	alt(K)	dalt(K)
9_{42}	0	2	1	1	1
10_{132}	-2	0	1	1	1
10_{136}	0	2	1	1	1
10_{139}	8	6	1	1	1
10_{145}	-4	-2	1	1	1
10_{152}	-8	-6	1	1	1
10_{154}	6	4	1	1	1
10_{161}	-6	-4	1	1	1
11_{n9}	6	4	1	1	1
11_{n12}	2	0	1	1	1
11_{n19}	-2	-4	1	1	1
11_{n20}	0	-2	1	1	1
11_{n24}	0	2	1	1	1
11_{n31}	4	2	1	1	1
11_{n38}	0	2	1	1	1
11_{n70}	2	4	1	1	1
11_{n77}	8	6	1	1	1
11_{n79}	0	2	1	1	1
11_{n92}	0	-2	1	1	1
11_{n96}	0	2	1	1	1
11_{n138}	0	2	1	1	1
11_{n183}	6	4	1	1	1

Table 2.1:

Chapter 3

The alternation number of a torus knot

3.1 Calculation of the signature of torus knots

We denote by $s(p,q)$ the Rasmussen s-invariant of the (p,q)-torus knot and by $\sigma(p,q)$ the negative signature of the (p,q)-torus knot. This notation is the same as Murasugi's ([43], p. 148). Under this notation, Corollary 2.4 states

$$\left|\frac{s(p,q)-\sigma(p,q)}{2}\right| \leq \mathrm{alt}(T_{p,q}). \tag{3.1}$$

In this section, we study the value $s(p,q)-\sigma(p,q)$ to estimate $\mathrm{alt}(T_{p,q})$ and prove Proposition 3.5. We can calculate the signature of torus knots by using the following theorem:

Theorem 3.1 ([22]).

I. Let us assume $q \leq 2p-1$. Then we have

- *(a) if p is an odd integer, $\sigma(p,q)+\sigma(2p-q,p)=p^2-1$,*
- *(b) if p is an even integer, $\sigma(p,q)+\sigma(2p-q,p)=p^2-2$.*

II. Let us assume $2p+1 \leq q$. Then we have

- *(a) if p is an odd integer, $\sigma(p,q)=\sigma(p,q-2p)+p^2-1$,* *(b) if p is an even integer, $\sigma(p,q) = \sigma(p,q-2p) + p^2$.*

III. We have $\sigma(p,q) = \sigma(q,p)$ and $\sigma(2,q) = q-1$.

We need the following three lemmas to prove Proposition 3.5.

Lemma 3.2. *We have*

$$\sigma(p,p+1) = \begin{cases} (p-1)(p+3)/2 & \text{if } p \text{ is odd,} \\ p^2/2 & \text{if } p \text{ is even.} \end{cases}$$

Proof. We prove by induction on p. It is easy to see that the lemma is true for the case $p = 2$. Suppose that the lemma is true for the case $p = n$ for a positive even integer n. We prove that the lemma is true for the case $p = n+1, n+2$. By applying (I) in Theorem 5.1 to $p = n+1$ and $q = n+2$, we have

$$\sigma(n+1, n+2) + \sigma(n, n+1) = (n+1)^2 - 1.$$

By the induction hypothesis, we have

$$\sigma(n+1, n+2) + n^2/2 = (n+1)^2 - 1.$$

Therefore, we have

$$\sigma(n+1, n+2) = n(n+4)/2. \tag{3.2}$$

Namely,

$$\sigma(n+1, n+2) = ((n+1)-1)((n+1)+3)/2.$$

This implies that the lemma is true for the case $p = n+1$. By using (I) in Theorem 5.1, we have

$$\sigma(n+2, n+3) + \sigma(n+1, n+2) = (n+2)^2 - 2,$$

by using the equality (3.2), we have

$$\sigma(n+2, n+3) + n(n+4)/2 = (n+2)^2 - 2,$$

therefore,

$$\sigma(n+2, n+3) = (n+2)^2/2.$$

This implies that the lemma is true for the case $p = n+2$. □

Lemma 3.3. *Let p be an odd integer with $p \geq 5$. Then we have*

$$4(p-1) \leq \sigma(p, p+2) \leq p^2 - 9. \tag{3.3}$$

Proof. The lemma is true for the case $p = 5$ because the signature of $T_{5,7}$ is 16. Suppose that the lemma is true for the case $p = n$ with $n \geq 5$. We first prove that the second inequality in (3.3) holds for the case $p = n + 2$. By using (I) in Theorem 5.1, we have

$$\sigma(n+2, n+4) + \sigma(n, n+2) = (n+2)^2 - 1. \tag{3.4}$$

By the induction hypothesis, we have

$$\begin{aligned}\sigma(n+2, n+4) &\leq (n+2)^2 - 1 - 4(n-1) \\ &= n^2 + 7 \\ &< (n+2)^2 - 9.\end{aligned}$$

This implies that the second inequality in (3.3) holds for the case $p = n + 2$. Next, we prove that the first inequality in (3.3) holds for the case $p = n + 2$. By the equality (3.4), we have

$$\sigma(n+2, n+4) = (n+2)^2 - 1 - \sigma(n, n+2).$$

By the induction hypothesis, we have

$$\begin{aligned}\sigma(n+2, n+4) &\geq (n+2)^2 - 1 - (n^2 - 9) \\ &= 4n + 12 \\ &> 4((n+2) - 1).\end{aligned}$$

This implies that the first inequality in (3.3) holds for the case $p = n+2$. □

Lemma 3.4. *Let us suppose $1 \leq m$ and $1 \leq i \leq 2p - 1$, where p and i are coprime. Then we have*

$$s(p, 2pm + i) = s(p, i) + 2pm(p-1),$$

$$\sigma(p, 2pm + i) = \begin{cases} \sigma(p, i) + m(p^2 - 1) & \text{if } p \text{ is odd,} \\ \sigma(p, i) + mp^2 & \text{if } p \text{ is even.} \end{cases}$$

Proof. It immediately follows from the equality (2.4) and (II) in Theorem 5.1. □

Proposition 3.5. *Let us assume* $3 \leq p$. *Then we have*

i. $s(p,q) - \sigma(p,q) = 0 \iff (p,q) = (3,4)$ *or* $(3,5)$.

ii. $s(p,q) - \sigma(p,q) \geq 4 \iff$ *otherwise.*

Proof. Recall p and q are coprime integers such that $3 \leq p < q$. We use this fact without mentioning explicitly. There are two cases to be considered:

Case 1. $q \leq 2p - 1$.
Case 2. $2p + 1 \leq q$.

We first consider Case 1. Set $q = p + i$, where $1 \leq i \leq p - 1$. This case is further divided into 1a: $i = 1$, 1b: $i = 2$ and p is odd, and 1c: $3 \leq i \leq p - 1$ and $4 \leq p$, where p and i are coprime.

Case 1a. When p is odd, by Lemma 5.2 we have

$$\begin{aligned} s(p,p+1) - \sigma(p,p+1) &= (p-1)p - (p-1)(p+3)/2 \\ &= (p-1)(p-3)/2. \end{aligned}$$

If $p = 3$, then we have

$$s(3,4) - \sigma(3,4) = 0.$$

If $p \geq 5$, then we have

$$s(p,p+1) - \sigma(p,p+1) \geq 4.$$

When p is even, by Lemma 5.2 we have

$$\begin{aligned} s(p,p+1) - \sigma(p,p+1) &= (p-1)p - p^2/2 \\ &= p(p-2)/2 \geq 4. \end{aligned}$$

Case 1b. If $p = 3$, then we have

$$s(3,5) - \sigma(3,5) = 0.$$

If $p \geq 5$, then by Lemma 5.3 we have

$$s(p, p+2) - \sigma(p, p+2) \geq (p-1)(p+1) - (p^2 - 9) = 8.$$

Case 1c. By using (I) in Theorem 5.1, we have

$$\begin{aligned} s(p, p+i) - \sigma(p, p+i) &\geq (p-1)(p+i-1) - (p^2 - 1) \\ &\geq (p-1)(p+2) - (p^2-1) = (p-1). \end{aligned}$$

If $p = 4$, then we have

$$s(p, p+i) - \sigma(p, p+i) \geq 3.$$

In fact, we have $s(p, p+i) - \sigma(p, p+i) \geq 4$ because the value $s(p, p+i) - \sigma(p, p+i)$ is an even integer. If $p \geq 5$, then we have

$$s(p, p+i) - \sigma(p, p+i) \geq 4.$$

Case 2. Set $q = 2pm + i$, where $1 \leq m$ and $1 \leq i \leq 2p - 1$. We note that p and i are coprime. If p is odd, by Lemma 5.4 we have

$$\begin{aligned} s(p, q) - \sigma(p, q) &= s(p, 2pm + i) - \sigma(p, 2pm + i) \\ &= s(p, i) - \sigma(p, i) + m(2p(p-1) - (p^2 - 1)) \\ &= s(p, i) - \sigma(p, i) + m(p-1)^2 \geq 4. \end{aligned}$$

If p is even, by Lemma 5.4 we have

$$\begin{aligned} s(p, q) - \sigma(p, q) &= s(p, i) - \sigma(p, i) + m(2p(p-1) - p^2) \\ &= s(p, i) - \sigma(p, i) + mp(p-2) \geq 8. \end{aligned}$$

□

3.2 Proofs of Theorems 3.6 and 3.7

Theorem 3.6. *We obtain the followings.*
(1) $\mathbf{alt}(T_{p,q}) = 0 \iff p = 2$.
(2) $\mathbf{alt}(T_{p,q}) = 1 \iff (p, q) = (3, 4)$ *or* $(3, 5)$.
(3) $\mathbf{alt}(T_{p,q}) \geq 2 \iff$ *otherwise.*

Proof. The assertion (i) is a well known fact. We consider the case $p \geq 3$. It is also well known that $T_{3,4}$ and $T_{3,5}$ are not alternating and we can deform $T_{3,4}$ and $T_{3,5}$ into an alternating knot by one crossing change (see also [24]). On the other hand, if $3 \leq p$ and $(p,q) \neq (3,4), (3,5)$, we have $2 \leq \mathrm{alt}(T_{p,q})$ by the equality. (3.1) and Proposition 3.5. □

Furthermore we obtain that many torus knots are indeed "far" from the alternating knots.

Theorem 3.7. *For each positive integer n, there exists a positive integer N such that for all $p > N$, we have $n \leq \mathrm{alt}(T_{p,q})$.*

Proof. It easily follows from the proof of Proposition 4.3. □

Corollary 3.8. *The almost alternating torus knots are just $T_{3,4}$ and $T_{3,5}$.*

Proof. It is well known that $T_{3,4}$ and $T_{3,5}$ are almost alternating [5]. Figure 3.1 also indicates $T_{3,4}$ and $T_{3,5}$ are almost alternating. On the other hand, if $3 \leq p$ and $(p,q) \neq (3,4), (3,5)$, we have $2 \leq \mathrm{alt}(T_{p,q}) \leq \mathrm{dalt}(T_{p,q})$ by Theorem 3.6 and the equality (1.2). This implies that $T_{p,q}$ is not almost alternating. □

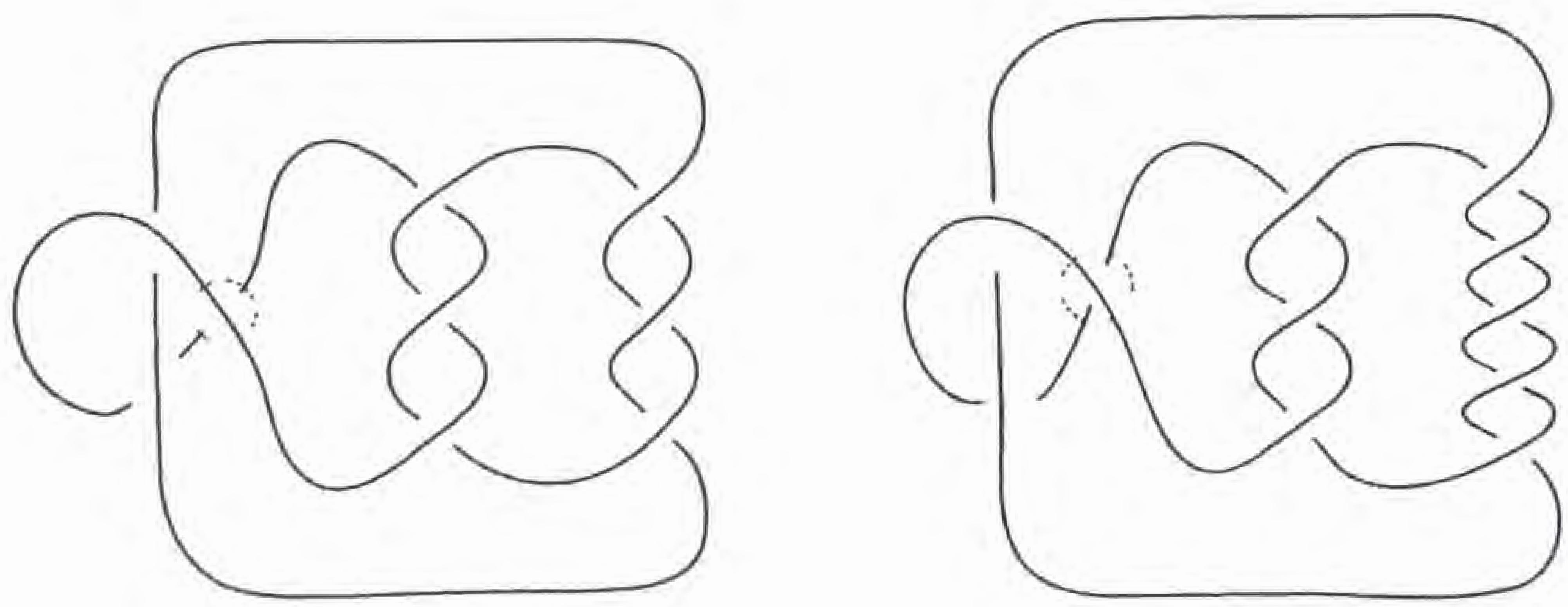

Figure 3.1: Diagrams of $T_{3,4}$ and $T_{3,5}$

3.3 Further argument

In this section, we furthermore study the lower bound for the alternation number of a torus knot. We have the following:

Proposition 3.9. *We have*

$$s(p,q) - \sigma(p,q) \equiv 0 \pmod 4.$$

Proposition 3.9 implies

$$2g(T_{p,q}) - \sigma(T_{p,q}) \equiv 0 \pmod 4$$

since we have $s(p,q) = 2g_*(T_{p,q}) = 2g(T_{p,q})$ [54]. Proposition 3.9 follows from the following four lemmas.

Lemma 3.10. *Let p and q be positive integers. If p and q are odd, $p \equiv 1 \pmod 4$ and q is even, or p is odd and $q \equiv 1 \pmod 4$, then we have*

$$s(p,q) \equiv 0$$

If $p \equiv 3 \pmod 4$ and q is even, or p is odd and $q \equiv 3 \pmod 4$, then we have

$$s(p,q) \equiv 2$$

Proof. It follows from the equality (2.4). □

In Lemmas 3.11 and 3.12, we do not assume $p < q$.

Lemma 3.11. *Let p and q be positive odd integers. Then we have*

$$\sigma(p,q) \equiv 0 \pmod 4.$$

Proof. The lemma is true for the case $p = 1$. Suppose that the lemma is true for the case $p = 1, 3, \cdots, n-2$, where $n-2$ is a positive odd integer. We show that $\sigma(n,q) \equiv 0 \pmod 4$ for any positive odd integer q. There are three cases to consider:

Case 1: $q < n$.
Case 2: $n < q \leq 2p - 1$.

Case 3: $2p+1 \leq q$.

Case 1. By using (III) in Theorem 5.1 and the hypothesis, we have

$$\sigma(n,q) = \sigma(q,n) \equiv 0 \pmod 4$$

Case 2. By using (I) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n,q) + \sigma(2n-q,n) &= n^2 - 1 \\ &\equiv 0 \pmod 4.\end{aligned}$$

Since we have $2n - q < n$, it follows that $\sigma(2n-q,n) \equiv 0$ by the hypothesis. This implies

$$\sigma(n,q) \equiv 0 \pmod 4.$$

Case 3. Set $q = 2nm + i$ where $1 \leq m$ and $1 \leq i \leq 2n-1$. By using (I) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n,q) &= \sigma(q-2n,n) + n^2 - 1 \\ &\equiv \sigma(q-2n,n) \pmod 4.\end{aligned}$$

By using (I) in Theorem 5.1 repeatedly, we have

$$\sigma(n,q) \equiv \sigma(i,n) \pmod 4.$$

Since we have $i \leq 2n-1$, by Cases 1 and 2 we have

$$\sigma(n,q) \equiv 0 \pmod 4.$$

This completes the proof. □

Lemma 3.12. *Let p be a positive odd integer, and q a positive even integer. Then we have*

$$\sigma(p,q) \equiv \begin{cases} 0 & \text{if } p \equiv 1 \pmod 4, \\ 2 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. The lemma is true for the case $p = 1$. Suppose that the lemma is true for the case $p = 1, \cdots, n-2$, where $n-2$ is a positive odd integer. First we consider the case $n - 2 \equiv 1 \pmod 4$. We show that

$$\sigma(n, q) \equiv 2 \pmod 4,$$

for any positive even integer q. This case is divided into three cases.

Case 1: $q < n$.
Case 2: $n < q \leq 2n - 1$.
Case 3: $2p + 1 \leq q$.

Case 1. If $n \leq 2q - 1$, by using (I) in Theorem 5.1 we have

$$\begin{aligned} \sigma(q, n) + \sigma(2q - n, q) &= q^2 - 2 \\ &\equiv 2 \pmod 4. \end{aligned}$$

Since we have $2q - n < n$ and $2q - n \equiv 1 \pmod 4$, by the hypothesis we have

$$\sigma(2q - n, q) \equiv 0 \pmod 4.$$

So we have

$$\sigma(q, n) \equiv 2 \pmod 4.$$

Namely,

$$\sigma(n, q) \equiv 2 \pmod 4.$$

If $2q + 1 \leq n$, by using (II) in Theorem 5.1 we have

$$\begin{aligned} \sigma(q, n) &= \sigma(n - 2q, q) + q^2 \\ &\equiv \sigma(n - 2q, q) \pmod 4. \end{aligned}$$

Since we have $n - 2q < n$ and $n - 2q \equiv 3 \pmod 4$, by the hypothesis we have

$$\sigma(n - 2q, q) \equiv 2 \pmod 4.$$

So we have

$$\sigma(q, n) \equiv 2 \pmod 4.$$

Namely,

$$\sigma(n,q) \equiv 2 \pmod 4.$$

Case 2. By using (I) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n,q) + \sigma(2n-q,n) &= n^2 - 1\\ &\equiv 0 \pmod 4.\end{aligned}$$

Since we have $2n - q < n$, by Case 1 we have

$$\sigma(n, 2n-q) \equiv 2 \pmod 4.$$

Namely,

$$\sigma(2n-q, n) \equiv 2 \pmod 4.$$

This implies

$$\sigma(n,q) \equiv 2 \pmod 4.$$

Case 3. Set $q = 2nm + i$ where $1 \leq m$ and $1 \leq i \leq 2n-1$. By using (II) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n,q) &= \sigma(q-2n,n) + n^2 - 1\\ &\equiv \sigma(q-2n,n) \pmod 4\\ &= \sigma(n,q-2n).\end{aligned}$$

By using (II) in Theorem 5.1 repeatedly, we have

$$\sigma(n,q) \equiv \sigma(n,i) \pmod 4.$$

By Cases 1 and 2, we have

$$\sigma(n,q) \equiv 2.$$

Next, we consider the case $n - 2 \equiv 3 \pmod 4$. We will show that

$$\sigma(n,q) \equiv 0 \pmod 4,$$

for any positive even integer q. This case is also divided into three cases.

Case 4: $q < n$.
Case 5: $n < q \leq 2n - 1$.
Case 6: $2p + 1 \leq q$.

Case 4. If $n \leq 2q - 1$, by using (I) in Theorem 5.1 we have

$$\begin{aligned}\sigma(q, n) + \sigma(2q - n, q) &= q^2 - 2 \\ &\equiv 2 \pmod{4}.\end{aligned}$$

Since we have $2q - n < n$ and $2q - n \equiv 3$, by the hypothesis we have

$$\sigma(2q - n, q) \equiv 2 \pmod{4}.$$

So we have

$$\sigma(n, q) \equiv 0 \pmod{4}.$$

If $2q + 1 \leq n$, by using (II) in Theorem 5.1 we have

$$\begin{aligned}\sigma(q, n) &= \sigma(n - 2q, q) + q^2 \\ &\equiv \sigma(n - 2q, q) \pmod{4}.\end{aligned}$$

Since we have $n - 2q < n$ and $n - 2q \equiv 1$, by the hypothesis we have

$$\sigma(n - 2q, q) \equiv 0 \pmod{4}.$$

So we have

$$\sigma(n, q) \equiv 0 \pmod{4}.$$

Case 5. By using (I) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n, q) + \sigma(2n - q, n) &= n^2 - 1 \\ &\equiv 0 \pmod{4}.\end{aligned}$$

Since we have $2n - q < n$, by Case 1 we have

$$\sigma(2n - q, n) \equiv 0 \pmod{4}.$$

This implies

$$\sigma(n, q) \equiv 0 \pmod{4}.$$

Case 6. Set $q = 2nm + i$, where $1 \leq m$ and $1 \leq i \leq 2n - 1$. By using (II) in Theorem 5.1, we have

$$\begin{aligned}\sigma(n,q) &= \sigma(q-2n,n) + n^2 - 1, \\ &\equiv \sigma(q-2n,n) \pmod 4. \\ &= \sigma(n,q-2n).\end{aligned}$$

By using (II) in Theorem 5.1 repeatedly, we have

$$\sigma(n,q) \equiv \sigma(n,i) \pmod 4$$

By Cases 4 and 5, we have

$$\sigma(n,q) \equiv 0 \pmod 4.$$

□

Lemma 3.13. *Let p be a positive even integer, and q a positive odd integer. Then we have*

$$\sigma(p,q) \equiv \begin{cases} 0 & \text{if } q \equiv 1 \pmod 4, \\ 2 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Proof. It follows from (III) in Theorem 3.1 and Lemma 3.12. □

3.4 Appendix

In this section, we study the lower bound for the alternation number of a torus knot by using Kauffman's signature formula of a torus knot.

For coprime intgers p and q, we define as follows.

$$f(a,b) = \frac{a}{p} + \frac{b}{q} + \frac{1}{2},$$

$$\epsilon(a,b) = \begin{cases} +1 & \text{if } 0 < f(a,b) < 1 \pmod 2, \\ \\ -1 & \text{if } 1 < f(a,b) < 2 \pmod 2, \end{cases}$$

$$\epsilon_+(a,b) = \begin{cases} +1 & \text{if } 0 < f(a,b) < 1 \pmod 2, \\ \\ 0 & \text{if } 1 < f(a,b) < 2 \pmod 2, \end{cases}$$

Kauffman [25] showed that

$$\sigma(T(p,q)) = \sum_{\substack{1 \le a \le p-1 \\ 1 \le b \le q-1}} \epsilon(a,b).$$

It is easy to check that

$$\frac{s(p,q) - \sigma(p,q)}{2} = \sum_{\substack{1 \le a \le p-1 \\ 1 \le b \le q-1}} \epsilon_+(a,b).$$

Now we give two examples.

Example 3.14. *Let p and q be the integers 7 and 10. Then $\sigma(T(p,q)) = -34$ and*

$$\frac{s(p,q)-\sigma(p,q)}{2} = \sum_{\substack{1\le a\le p-1\\1\le b\le q-1}} \epsilon_+(a,b) = 10.$$

6	$\frac{102}{70}$	$\frac{109}{70}$	$\frac{116}{70}$	$\frac{123}{70}$	$\frac{130}{70}$	$\frac{137}{70}$	$\frac{144}{70}$	$\frac{151}{70}$	$\frac{158}{70}$
5	$\frac{92}{70}$	$\frac{99}{70}$	$\frac{106}{70}$	$\frac{113}{70}$	$\frac{120}{70}$	$\frac{127}{70}$	$\frac{134}{70}$	$\frac{141}{70}$	$\frac{148}{70}$
4	$\frac{82}{70}$	$\frac{89}{70}$	$\frac{96}{70}$	$\frac{103}{70}$	$\frac{110}{70}$	$\frac{117}{70}$	$\frac{124}{70}$	$\frac{131}{70}$	$\frac{138}{70}$
3	$\frac{72}{70}$	$\frac{79}{70}$	$\frac{86}{70}$	$\frac{93}{70}$	$\frac{100}{70}$	$\frac{107}{70}$	$\frac{114}{70}$	$\frac{121}{70}$	$\frac{128}{70}$
2	$\frac{62}{70}$	$\frac{69}{70}$	$\frac{76}{70}$	$\frac{83}{70}$	$\frac{90}{70}$	$\frac{97}{70}$	$\frac{104}{70}$	$\frac{111}{70}$	$\frac{118}{70}$
1	$\frac{52}{70}$	$\frac{59}{70}$	$\frac{66}{70}$	$\frac{73}{70}$	$\frac{80}{70}$	$\frac{87}{70}$	$\frac{94}{70}$	$\frac{101}{70}$	$\frac{108}{70}$
$f(a,b)$	1	2	3	4	5	6	7	8	9

6	0	0	0	0	0	0	+1	+1	+1
5	0	0	0	0	0	0	0	+1	+1
4	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
2	+1	+1	0	0	0	0	0	0	0
1	+1	+1	+1	0	0	0	0	0	0
$\epsilon_+(a,b)$	1	2	3	4	5	6	7	8	9

Example 3.15. *Let p and q be the integers* 8 *and* 11. *Then* $\sigma(T(p,q)) = -52$ *and*

$$\frac{s(p,q) - \sigma(p,q)}{2} = \sum_{\substack{1 \le a \le p-1 \\ 1 \le b \le q-1}} \epsilon_+(a,b) = 14.$$

7	$\frac{129}{88}$	$\frac{137}{88}$	$\frac{145}{88}$	$\frac{153}{88}$	$\frac{161}{88}$	$\frac{169}{88}$	$\frac{177}{88}$	$\frac{185}{88}$	$\frac{193}{88}$	$\frac{201}{88}$
6	$\frac{118}{88}$	$\frac{126}{88}$	$\frac{134}{88}$	$\frac{142}{88}$	$\frac{150}{88}$	$\frac{158}{88}$	$\frac{166}{88}$	$\frac{174}{88}$	$\frac{182}{88}$	$\frac{190}{88}$
5	$\frac{107}{88}$	$\frac{115}{88}$	$\frac{123}{88}$	$\frac{131}{88}$	$\frac{139}{88}$	$\frac{147}{88}$	$\frac{155}{88}$	$\frac{163}{88}$	$\frac{171}{88}$	$\frac{179}{88}$
4	$\frac{96}{88}$	$\frac{104}{88}$	$\frac{112}{88}$	$\frac{120}{88}$	$\frac{128}{88}$	$\frac{136}{88}$	$\frac{144}{88}$	$\frac{152}{88}$	$\frac{160}{88}$	$\frac{168}{88}$
3	$\frac{85}{88}$	$\frac{93}{88}$	$\frac{101}{88}$	$\frac{109}{88}$	$\frac{117}{88}$	$\frac{125}{88}$	$\frac{133}{88}$	$\frac{141}{88}$	$\frac{149}{88}$	$\frac{157}{88}$
2	$\frac{74}{88}$	$\frac{82}{88}$	$\frac{90}{88}$	$\frac{98}{88}$	$\frac{106}{88}$	$\frac{114}{88}$	$\frac{122}{88}$	$\frac{130}{88}$	$\frac{138}{88}$	$\frac{146}{88}$
1	$\frac{63}{88}$	$\frac{71}{88}$	$\frac{79}{88}$	$\frac{87}{88}$	$\frac{95}{88}$	$\frac{103}{88}$	$\frac{111}{88}$	$\frac{119}{88}$	$\frac{127}{88}$	$\frac{135}{88}$
$f(a,b)$	1	2	3	4	5	6	7	8	9	10

7	0	0	0	0	0	0	+1	+1	+1	+1
6	0	0	0	0	0	0	0	0	+1	+1
5	0	0	0	0	0	0	0	0	0	+1
4	0	0	0	0	0	0	0	0	0	0
3	+1	0	0	0	0	0	0	0	0	0
2	+1	+1	0	0	0	0	0	0	0	0
1	+1	+1	+1	+1	0	0	0	0	0	0
$\epsilon_+(a,b)$	1	2	3	4	5	6	7	8	9	10

14	-	-	-	-	-	-	-	-	-	-	-	42
13	-	-	-	-	-	-	-	-	-	-	30	36
12	-	-	-	-	-	-	-	-	-	30	-	-
11	-	-	-	-	-	-	-	-	20	24	26	30
10	-	-	-	-	-	-	-	20	-	22	-	-
9	-	-	-	-	-	-	12	16	-	20	20	-
8	-	-	-	-	-	12	-	14	-	16	-	18
7	-	-	-	-	6	8	10	10	12	12	-	18
6	-	-	-	6	-	-	-	8	-	12	-	-
5	-	-	2	4	4	4	-	8	8	8	10	-
4	-	2	-	2	-	4	-	4	-	6	-	6
3	0	0	-	2	2	-	2	2	-	4	4	-
$\frac{s(p,q)-\sigma(p,q)}{2}$	4	5	6	7	8	9	10	11	12	13	14	15

Lemma 3.16.

$$\epsilon(a,b) = \epsilon(p-a, q-b).$$

Proof. The proof is followed from two claims.
Claim. $\epsilon(a,b) = \epsilon(-a,-b)$.
This follows from the following.

$$0 < \frac{a}{p} + \frac{b}{q} + \frac{1}{2} < 1 \mod 2 \tag{3.5}$$

$$\Longleftrightarrow 0 > -\frac{a}{p} - \frac{b}{q} - \frac{1}{2} > -1 \mod 2 \tag{3.6}$$

$$\Longleftrightarrow 1 > -\frac{a}{p} - \frac{b}{q} + \frac{1}{2} > 0 \mod 2. \tag{3.7}$$

Claim. $\epsilon(a,b) = \epsilon(p+a, q+b)$.
This follows from the following.

$$0 < \frac{a}{p} + \frac{b}{q} + \frac{1}{2} < 1 \mod 2 \tag{3.8}$$

$$\Longleftrightarrow 2 < \frac{p+a}{p} + \frac{q+b}{q} + \frac{1}{2} < 3 \mod 2. \tag{3.9}$$

□

Corollary 3.17.

$$\frac{s(p,q) - \sigma(p,q)}{2} = 0 \mod 2.$$

Lemma 3.18. *Let p and q be coprime positive intergers with $3 \leq p < q$.*
(1) If $\epsilon(1,1) = -1$, then $(p,q) = (3,4)$ or $(3,5)$.
(2) If $\epsilon(1,2) = -1$, then $(p,q) = (3,4), (3,5), (3,7), (3,8), (3,10), (3,11), (4,5), (4,7)$ or $(5,6)$.

Proof. Note that $0 < f(1,1) < 2$ and $0 < f(1,2) < 2$.
(1) It suffices to find (p,q) such that $1 < f(1,1)$. If $4 \leq p$, then

$$f(1,1) = \frac{1}{p} + \frac{1}{q} + \frac{1}{2} < \frac{2}{p} + \frac{1}{2} \tag{3.10}$$

$$\leq 1. \tag{3.11}$$

Therefore there is no (p,q) such that $1 < f(1,1)$. If $p=3$, then

$$f(1,1) = \frac{1}{p} + \frac{1}{q} + \frac{1}{2} = \frac{1}{q} + \frac{5}{6}. \tag{3.12}$$

We obtain $(3,4)$ and $(3,5)$ such that $1 < f(1,1)$.
(2) It suffices to find (p,q) such that $1 < f(1,2)$. If $p \geq 6$, then

$$f(1,2) = \frac{1}{p} + \frac{2}{q} + \frac{1}{2} < \frac{3}{p} + \frac{1}{2} \tag{3.13}$$

$$\leq 1. \tag{3.14}$$

Therefore there is no (p,q) such that $1 < f(1,2)$. If $p=5$, then

$$f(1,2) = \frac{1}{5} + \frac{2}{q} + \frac{1}{2} = \frac{2}{q} + \frac{7}{10}. \tag{3.15}$$

We obtain $(5,6)$ such that $1 < f(1,2)$. If $p=4$, then

$$f(1,2) = \frac{1}{4} + \frac{2}{q} + \frac{1}{2} = \frac{2}{q} + \frac{3}{4}. \tag{3.16}$$

We obtain $(4,5)$ $(4,7)$ such that $1 < f(1,2)$. If $p=3$, then

$$f(1,2) = \frac{1}{3} + \frac{2}{q} + \frac{1}{2} = \frac{2}{q} + \frac{5}{6}. \tag{3.17}$$

We obtain $(3,4)$, $(3,5)$, $(3,7)$, $(3,8)$, $(3,10)$ and $(3,11)$ such that $1 < f(1,2)$. □

Lemma 3.19. *Let p and q be coprime positive intergers with $4 \leq p < q$.*
(1) For $1 \leq a \leq [\frac{p}{q}(\frac{q}{2}-1)]$, $\epsilon(a,1) = +1$.
(2) For $1 \leq b \leq [\frac{q}{p}(\frac{p}{2}-1)]$, $\epsilon(1,b) = +1$,
where the symbol $[x]$ means the greatest integer less than or equal to x.

Proof. (1) We obtain the following.

$$0 < \frac{a}{p} + \frac{1}{q} + \frac{1}{2} < 1 \tag{3.18}$$

$$\Longleftrightarrow -\frac{pq}{2} < aq + p < \frac{pq}{2} \tag{3.19}$$

$$\Longleftrightarrow -\frac{p}{q}(\frac{q}{2}+1) < a < \frac{p}{q}(\frac{q}{2}-1). \tag{3.20}$$

(2) We obtain the following.

$$\frac{1}{p}+\frac{b}{q}+\frac{1}{2}<1 \tag{3.21}$$

$$\Longleftrightarrow \frac{pq}{2}<q+bp<\frac{pq}{2} \tag{3.22}$$

$$\Longleftrightarrow -\frac{q}{p}(\frac{p}{2}+1)<b<\frac{q}{p}(\frac{p}{2}-1). \tag{3.23}$$

□

Theorem 3.20. *For each* n, *there exist finitely many* (p,q) *such that* $\mathrm{alt}(T(p,q)) = n$.

Theorem 3.21. *We obtain the following.*
(1) $\mathrm{alt}(T_{p,q})=0 \Longleftrightarrow p=2$.
(2) $\mathrm{alt}(T_{p,q})=1 \Longleftrightarrow (p,q)=(3,4)$ *or* $(3,5)$.
(3) $\mathrm{alt}(T_{p,q})=2 \Longrightarrow (p,q)=(3,7),(3,8),(3,10),(3,11),(4,5),\ (4,7)$ *or* $(5,6)$.
(4) $\mathrm{alt}(T_{p,q})\geq 4 \Longleftarrow$ *otherwise.*

Note that T. Kanenobu determined in [24] the alternation number of some torus knots by using our method,

$$\mathrm{alt}(T_{3,6m+i})=2m, \quad \mathrm{alt}(T_{4,5})=2,$$

where m is a positive integer and $i=1$ or 2.

Chapter 4

The alternation number of a closed 3-braid

4.1 An upper bound for the dealternating number of a closed 3-braid

In this section, we recall an upper bound for the dealternating number of a closed 3-braid. The *n-braid group* B_n, $n \in \mathbb{Z}_{>0}$, is a group which has the following presentation.

$$\left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_t\sigma_s = \sigma_s\sigma_t & (|t-s| > 1) \\ \\ \sigma_t\sigma_s\sigma_t = \sigma_s\sigma_t\sigma_s & (|t-s| = 1) \end{array} \right\rangle .$$

Let β be an n-braid of the form

$$\prod_{j=1}^{m} \sigma_{i_j}^{a_j},$$

where $i_j \neq i_{j+1}$ and a_j is a nonzero integer for $j = 1, 2, \dots, m$. An n-braid β is *positive* if a_j is positive for all j. We note that a positive 3-braid is conjugate to a braid of the form

$$\prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i},$$

where $p_i,\ q_i \in \mathbb{Z}_{>0}$ for $i = 1, 2, \ldots, r$. We denote by $\widehat{\beta}$ the closure of a braid β.

Lemma 4.1 ([2]). *Let β be a positive 3-braid of the form*

$$\prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i},$$

where $p_i,\ q_i \in \mathbb{Z}_{>0}$ for $i = 1, 2, \ldots, r$. Then we have

$$\mathrm{dalt}(\widehat{\beta}) \leq r - 1.$$

4.2 The alternation number and the dealternating number of a closed positive 3-braid

In this section, we determine the alternation numbers and the dealternating numbers of some closed positive 3-braid knots.

We recall the following two lemmas.

Lemma 4.2 ([54]). *Let D be a positive diagram of a positive knot K. Then we have*

$$s(K) = c(D) - o(D) + 1.$$

Set $\Delta = \sigma_1 \sigma_2 \sigma_1$.

Lemma 4.3 ([2]). *Let β be a positive 3-braid of the form*

$$\Delta^{2n} \prod_{i=1}^{r} \sigma_1^{p_i} \sigma_2^{q_i}$$

such that $\widehat{\beta}$ is a knot, $n \geq 0$ and $p_i, q_i \geq 2$ for $i = 1, 2, \ldots, r$. Then we have

$$-\sigma(\widehat{\beta}) = 4n - 2r + \sum_{i=1}^{r} (p_i + q_i).$$

The following is main result in this chapter.

Theorem 4.4. *Let β be a positive 3-braid of the form*

$$\Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}$$

such that $\widehat{\beta}$ is a knot, $n \geq 0$ and $p_i, q_i \geq 2$ for $i = 1, 2, \ldots, r$. Then we have

$$\mathrm{alt}(\widehat{\beta}) = \mathrm{dalt}(\widehat{\beta}) = n + r - 1.$$

Proof. First we show that $\mathrm{dalt}(\widehat{\beta}) \leq n + r - 1$.

We modify β into $\sigma_2(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}$ by the following equalities.

$$\begin{aligned}
\beta &\doteq \Delta^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}\\
&= (\sigma_2\sigma_1^2\sigma_2)^n\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}\\
&= \sigma_2(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}.
\end{aligned}$$

If $r = 1$, we modify β into $(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1+1}$ by the following equalities.

$$\begin{aligned}
\beta &= \sigma_2(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n}\sigma_1^{p_1}\sigma_2^{q_1}\\
&= (\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1+1}.
\end{aligned}$$

If $r \geq 2$, we modify β into $(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1}\{\prod_{i=2}^{r-1}\sigma_1^{p_i}\sigma_2^{q_i}\}\sigma_1^{p_r}\sigma_2^{q_r+1}$ by the following equalities.

$$\begin{aligned}
\beta &= \sigma_2(\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n}\prod_{i=1}^{r}\sigma_1^{p_i}\sigma_2^{q_i}\\
&= (\sigma_1^2\sigma_2^2)^{n-1}\sigma_1^2\sigma_2\sigma_1^{2n+p_1}\sigma_2^{q_1}\{\prod_{i=2}^{r-1}\sigma_1^{p_i}\sigma_2^{q_i}\}\sigma_1^{p_r}\sigma_2^{q_r+1}.
\end{aligned}$$

By Lemma 4.1, we have

$$\mathrm{dalt}(\widehat{\beta}) \leq n + r - 1.$$

Next we show that $\mathrm{alt}(\widehat{\beta}) \geq n + r - 1$. Lemmas 4.2 and 4.3 imply that

$$\begin{aligned} s(\widehat{\beta}) &= 6n - 2 + \sum_{i=1}^{r}(p_i + q_i), \\ -\sigma(\widehat{\beta}) &= 4n - 2r + \sum_{i=1}^{r}(p_i + q_i). \end{aligned}$$

By Corollary 2.4, we have

$$\frac{1}{2}|s(K) - (-\sigma(K))| = n + r - 1 \leq \mathrm{alt}(\widehat{\beta}).$$

Therefore, we have

$$\mathrm{alt}(\widehat{\beta}) = \mathrm{dalt}(\widehat{\beta}) = n + r - 1.$$

□

Chapter 5

The alternation number of a Montesinos knot

In this Chapter, we show that non-alternating Montesinos links and semi-alternating links are almost alternating. Throughout this section, we assume that tangles and tangle diagrams have four ends. We recall some notations for tangle diagrams.

Two tangle diagrams T and T' are *equivalent*, denoted by $T \sim T'$, if they are related by a finite sequence of the Reidemeister moves. The *integral tangle diagram* $[n]$ ($n \in \mathbb{Z}$) is n horizontal half twists (see Figure 5.1).

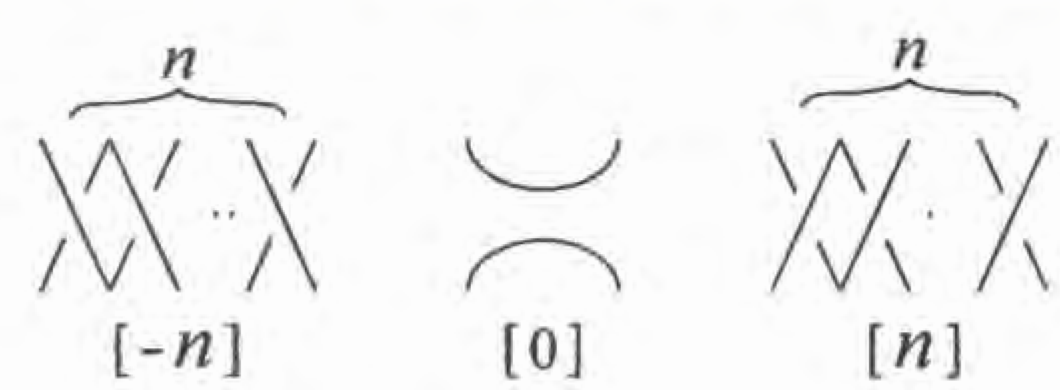

Figure 5.1: The integral tangle diagrams

The *sum* (resp. the *product*) of two tangle diagrams T and S is the tangle diagram as in Figure 5.2. We denote the sum (resp. the product) of two tangle diagrams T and S by $T + S$ (resp. $T * S$).

A *rational tangle diagram* is a tangle diagram obtained from integral tangle diagrams using only the operation of product.

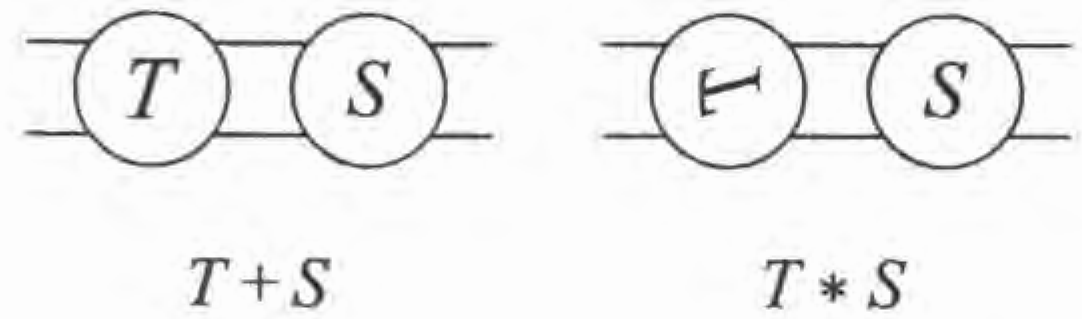

Figure 5.2: The sum and the product of tangle diagrams T and S

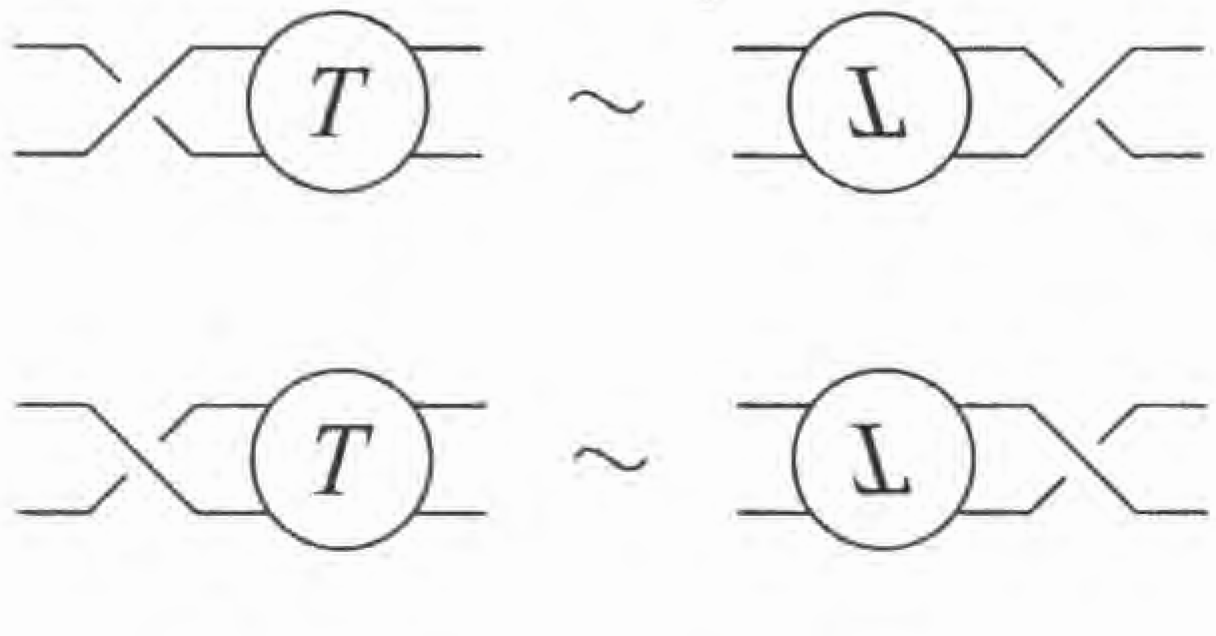

Figure 5.3: A flype

Remark 5.1. Let $a_1, a_2, a_3, \ldots, a_n$ be integers. A rational tangle diagram $(\cdots(([a_1] * [a_2]) * [a_3]) * \cdots) * [a_n]$ is alternating if and only if $a_1, a_2, a_3, \ldots, a_n$ have the same sign. For any rational tangle diagram T, there exists an alternating rational tangle diagram T' such that $T \sim T'$,

A *flype* is a local move on a tangle diagram (or a link diagram) as in Figure 5.3.

Remark 5.2. Let T be a rational tangle diagram and s an integer. By flypes, we have $T + [s] \sim [s] + T$ (see [26]).

We depict a symbol o (resp. u) near each end point of a tangle diagram as in Figure 5.4 if an over-crossing (resp. an under-crossing) appears first when we traverse the component from the end point. We call a tangle diagram on the left side (resp. the right side) in Figure 5.4 of *type 1* (resp. of *type 2*). We note that an alternating tangle diagram is of type 1 or of type 2.[1] For example, the integral tangle [1] is of type 1, and [−1] is of type 2. We give two lemmas needed later.

[1]Use a checkerboard coloring for the diagram.

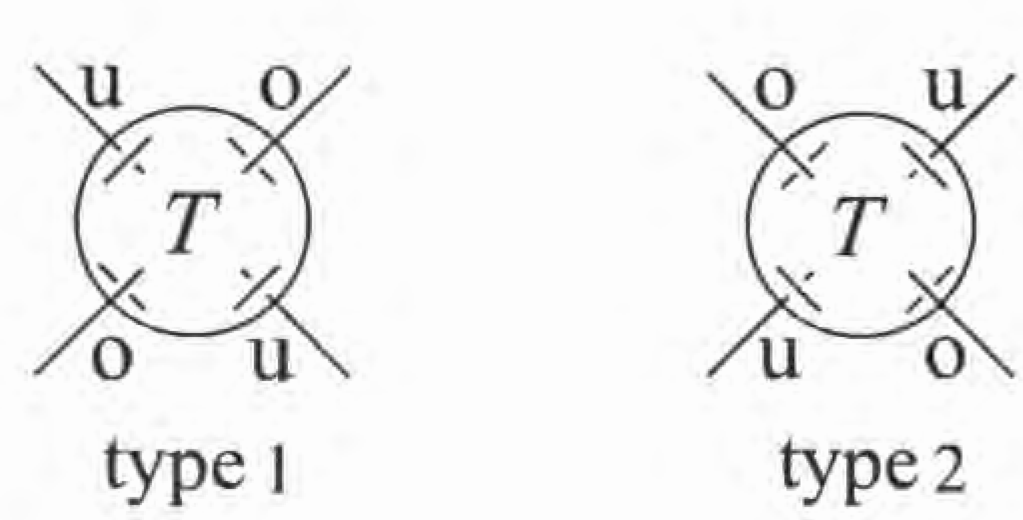

Figure 5.4: Tangles of type 1 and of type 2

Lemma 5.3. *Let T be an alternating tangle diagram, and R an alternating rational tangle diagram. Then there exist an alternating tangle T' and an integer r such that $T + R \sim T' + [r]$.*

Proof. If T and R are both of type 1 or of type 2, then we set $T' = T + R$ and $r = 0$. We assume that the tangle diagram T is of type 1, and the rational tangle diagram R is of type 2. In this case, R is represented by $(\cdots([-a_1] * [-a_2]) * \cdots) * [-a_n]$, where $a_1, a_2, \ldots, a_n$ are positive integers. If R is an integral tangle diagram $[r]$, then we set $T' = T$ and $[r] = R$. If R is not an integral tangle diagram, we obtain the sum of an alternating tangle diagram of type 1 and the integral tangle diagram $[-(a_n + 1)]$ by deforming $T + R$ as in Figure 5.5. If T is of type 2 and R is of type 1, then the proof is reduced to the case treated above by taking the mirror image of $T + R$. □

Lemma 5.4. *Let R_i $(i = 1, \ldots, m)$ be alternating rational tangle diagrams. Then there exist an alternating tangle diagram T and an integer s such that $R_1 + \cdots + R_m \sim T + [s]$.*

Proof. We prove this lemma by induction on m. It is trivial for the case $m = 1$. The lemma is also true by Lemma 5.3 for the case $m = 2$.

Suppose that the lemma is true for the case $m = n$ such that $n \geq 2$. We show that the lemma is true for the case $m = n+1$. By the assumption, there exist an alternating tangle diagram T and $s \in \mathbb{Z}$ such that $R_1 + \cdots + R_n \sim T + [s]$. By Remark 5.2, we have

$$R_1 + \cdots + R_{n+1} \sim T + R_{n+1} + [s].$$

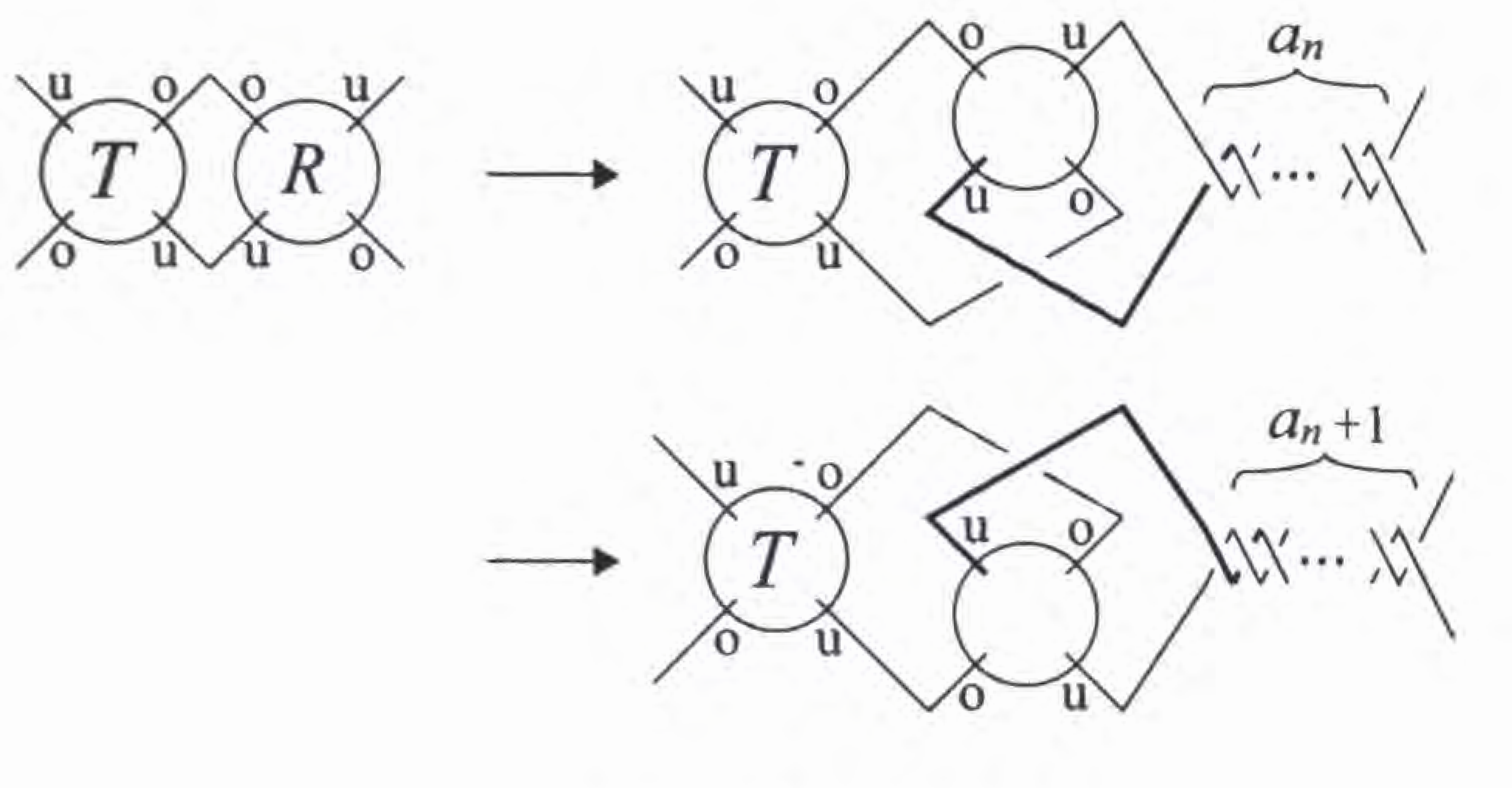

Figure 5.5:

By Lemma 5.3, there exist an alternating tangle diagram T' and $r \in \mathbb{Z}$ such that $T + R_{n+1} \sim T' + [r]$. We obtain the sum of an alternating tangle diagram T' and an integral tangle diagram $[r + s]$. □

The *numerator* $N(T)$ and the *denominator* $D(T)$ of a tangle diagram T are the closures of T as in Figure 5.6. A *Montesinos link* is a link which has a diagram represented by $N(R_1 + \cdots + R_m)$, where $R_1, \ldots, R_m$ are rational tangle diagrams (see Figure 5.7).

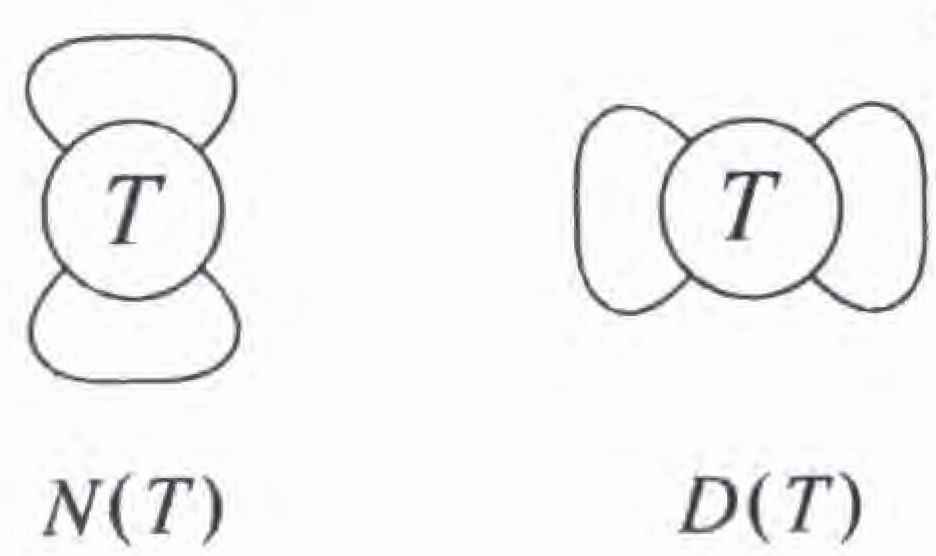

Figure 5.6: The numerator and the denominator of a tangle diagram T

Proposition 5.5. *Montesinos links are alternating or almost alternating.*

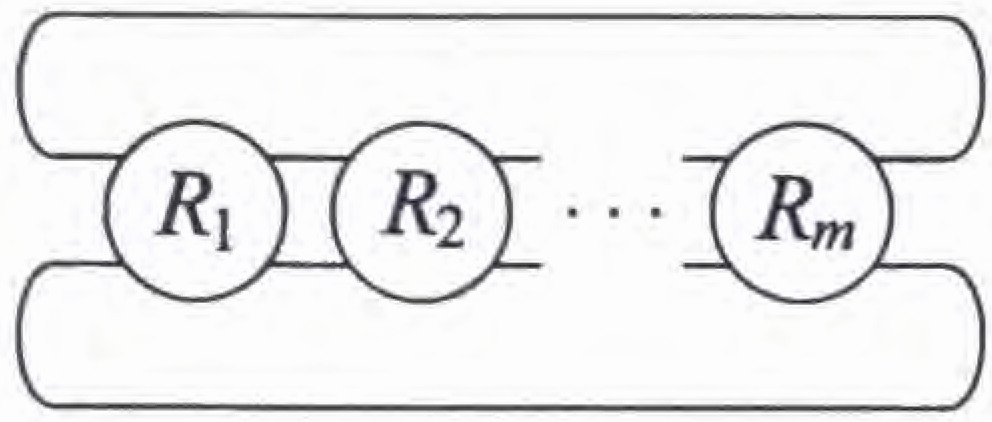

Figure 5.7: A diagram of a Montesinos link

Proof. By Remark 5.1, every Montesinos link has a diagram D represented by $N(R_1 + \cdots + R_m)$, where $R_1, \ldots, R_m$ are alternating rational tangle diagrams. Then, by Lemma 5.4, the diagram D can be deformed into a diagram represented by the numerator $N(T + [s])$ of the sum of an alternating tangle diagram T and an integral tangle diagram $[s]$.

If T and $[s]$ are both of type 1 or of type 2, then $N(T + [s])$ is alternating. If T is of type 1 and $[s]$ is of type 2, then we deform $N(T + [s])$ into an almost alternating diagram as in Figure 5.8. If T is of type 2 and $[s]$ is of type 1, then the proof is reduced to the case treated above by taking the mirror image of the diagram.

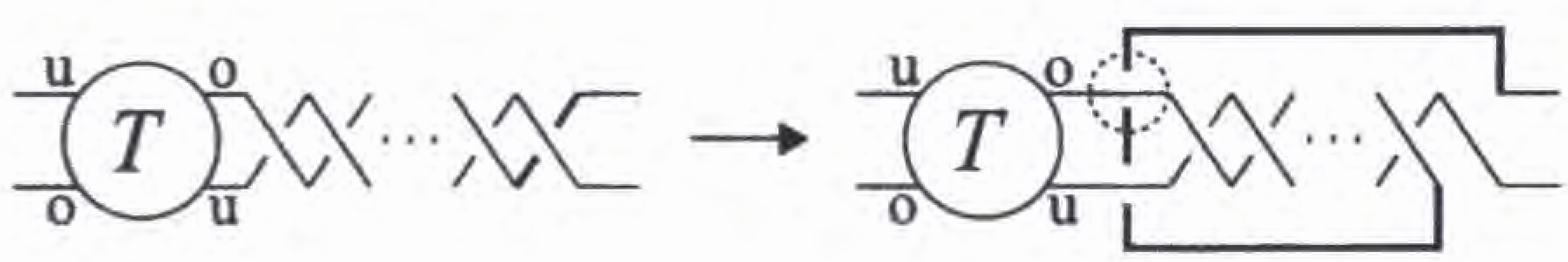

Figure 5.8:

□

A tangle diagram T is *strongly alternating* if both $N(T)$ and $D(T)$ are reduced alternating. A link L is *semi-alternating* if L has a non-alternating diagram which is represented by $N(T + T')$, where T and T' are strongly alternating tangle diagrams (see Figure 5.9).

Proposition 5.6. *Semi-alternating links are almost alternating.*

Proof. Lickorish and Thistlethwaite [35] showed that semi-alternating links are non-alternating. On the other hand, we can obtain an almost alternating

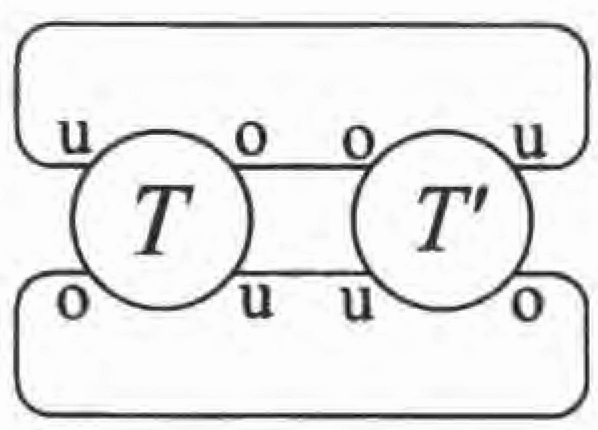

Figure 5.9: A diagram of a semi-alternating link

diagram of a semi-alternating link as in Figure 5.10. □

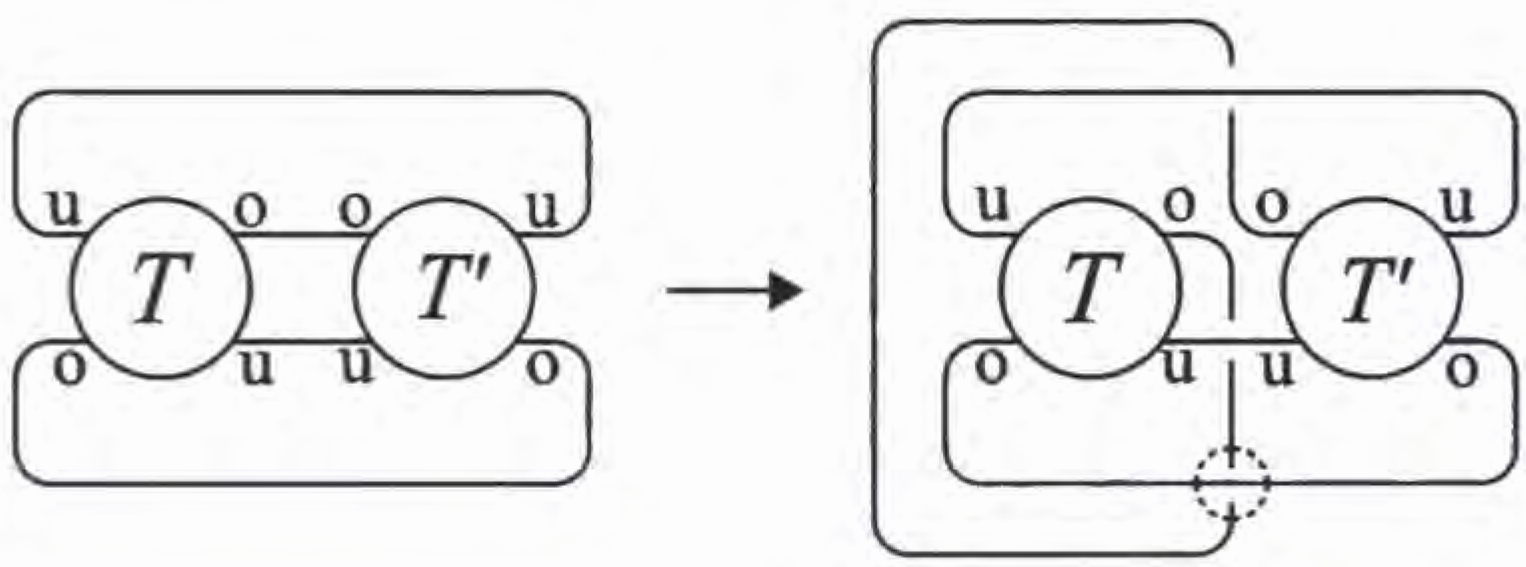

Figure 5.10:

For a non-split link L, Champanerkar and Kofman [12] showed

$$\omega_{Kh}(L) - 2 \leq \mathrm{dalt}(L).$$

Propositions 5.5 and 5.6 imply the following.

Corollary 5.7. *Let L be a Montesinos link or a semi-alternating link. Then*

$$\omega_{Kh}(L) \leq 3.$$

Recall that A. Champanerkar and I. Kofman give infinitely many pretzel links with $\omega_{Kh}(L) = 3$ [13].

Corollary 5.8. *Let L be a Montesinos knot or a semi-alternating knot. Then*

$$|s(K) - (-\sigma(K))| \leq 2.$$

Chapter 6

The Turaev genus of an adequate knot

6.1 Background of the Turaev genus and an adequate knot

The Jones polynomial provided a solution of the Tait conjecture, which states that a reduced alternating link digram has minimal crossing number [27], [47] and [61]. Furthermore, Murasugi [47] and Thistlethwaite [61] proved that a minimal crossing diagram of a prime alternating link is reduced alternating. To give a simple proof of these results, Turaev [62] introduced a surface associated to a connected link diagram, now called the Turaev surface associated to the diagram. We denote by $g_T(D)$ the genus of the Turaev surface associated to a connected link diagram D. Dasbach et al. [16] introduced the notion of the Turaev genus $g_T(L)$ of a non-split link L which is defined to be the minimal number of $g_T(D)$ associated to diagrams D of the link L. They showed that a non-split link L is alternating if and only if $g_T(L) = 0$. In this sense, the Turaev genus of a non-split link is an obstruction to the link being alternating.

In general, it is difficult to determine the Turaev genus of a non-split link. However, many non-alternating links are known to be of Turaev genus one. For example, non-split almost alternating links which include non-alternating

Figure 6.1: an A-splice and a B-splice

knots of eleven or fewer crossings except 11_{n95} and 11_{n118}, non-alternating Montesinos links and semi-alternating links are of Turaev genus one [2].

Lickorish and Thistlethwaite [35] introduced the concept of an adequate link, which is a generalization of an alternating link defined as follows. Let L be a link, D a link diagram of L and $c_1, \cdots, c_n$ the crossings of D. A *state* for the diagram D is a function $s : \{c_1, c_2, \cdots, c_n\} \to \{1, -1\}$. An *A-splice* and a *B-splice* are local moves on a link diagram as in Figure 6.1. We denote by sD the loops which are obtained from D by applying A-splices at crossings c such that $s(c) = 1$ and by applying B-splices at crossings c' such that $s(c') = -1$. We denote by $|sD|$ the number of components of sD. Let s_A be the state such that $s_A(c_i) = 1$ for every i and s_B the state such that $s_B(c_i) = -1$ for every i.

A link diagram D is said to be *adequate* [35] if $|s_A D| > |sD|$ for every state s such that $\sum_{i=1}^{n} s(c_i) = n - 2$ and $|s_B D| > |sD|$ for every state s such that $\sum_{i=1}^{n} s(c_i) = 2 - n$. A typical adequate diagram is a reduced alternating diagram [35] (see also Proposition 5.3 in [34]). Another example is an *n-semi-alternating diagram* which was introduced by Beltrami [9]. An adequate diagram also has minimal crossing number. This result was first proved in [61] using the Kauffman polynomial. Simpler proofs were given in [34] using the Jones polynomial and in [30] using the Khovanov homology. An *adequate link* is a link which admits an adequate diagram of the link. Note that it is known from [61] that a minimal crossing diagram of an adequate link is also adequate.

The *genus* of a knot is the minimal genus of any connected compact oriented surface whose boundary is the knot. It is well known that the genus of a knot is additive under connected sum. A remarkable property of the genus is that it may distinguish a knot from a mutant of the original knot. For example, the genus distinguishes the Terasaka-Kinoshita knot from the

Conway knot (e.g., see [5]). A mutant of a link (or a diagram of a link) is defined in the next section. In this chapter, we show

Theorem 6.10. *Let K be a knot which admits an adequate diagram D. Then*

$$g_T(K) = g_T(D) = 1/2(c(D) - |s_A D| - |s_B D|) + 1 = \omega_{Kh}(K) - 2 = c(K) - \operatorname{span} V_K(t). \tag{6.1}$$

Here we denote by $c(L)$ the crossing number of a link L, by $\omega_{Kh}(L)$ the homological width of the Khovanov homology of L and by $\operatorname{span} V_L(t)$ the span of the Jones polynomial of L. For the definition of the homological width of the Khovanov homology, see [12]. By the equation (6.1), we obtain the additivity of the Turaev genus of adequate knots.

Corollary 6.13. *Let K and K' be adequate knots. Then*

$$g_T(K \# K') = g_T(K) + g_T(K').$$

The following corollary suggests that the Turaev genus of a knot might be preserved under mutation.

Corollary 6.14. *Let K be a knot admitting an adequate diagram D and K' a knot which has a mutant of D. Then*

$$g_T(K) = g_T(K').$$

Note that we can choose so that K is the Kinoshita-Terasaka knot and K' is the Conway knot as a special case of Corollary 6.14. For more details, see Section 6.4. We also show that an n-semi-alternating knot is of Turaev genus n (Theorem 6.19). This is the first example of a class which contains adequate knots K with $g_T(K) \geq 2$.

6.2 Preliminary

In this section, we recall several definitions and results need later.

The Jones polynomial $V_L(t)$ of a link L is a Laurent polynomial in $t^{1/2}$ with integer coefficients (e.g., [34]). We denote by $\operatorname{span} V_L(t)$ the span of $V_L(t)$, that is, the difference between the maximal degree of $t^{1/2}$ and the minimal degree of $t^{1/2}$ that occur in $V_L(t)$. Then

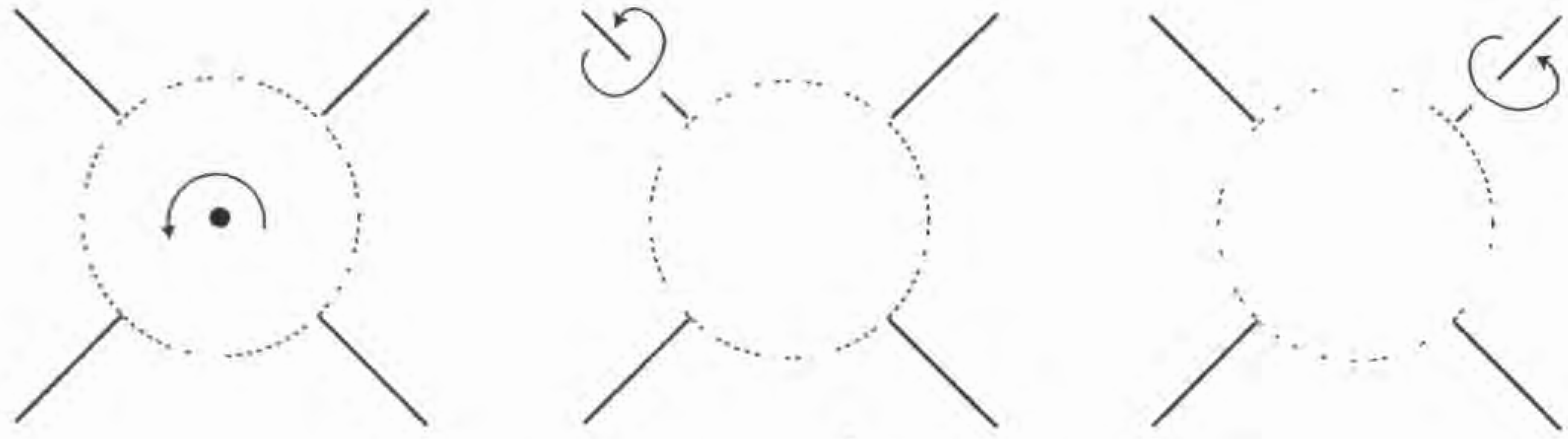

Figure 6.2: a mutation of a link (a diagram)

Proposition 6.1 ([35]). *Let L be a link which admits a connected adequate diagram D. Then*

$$\operatorname{span} V_L(t) = \frac{1}{2}(c(D) + |s_A D| + |s_B D| - 2).$$

A *mutation* is an operation on a link (resp. a diagram) that can produce different links (resp. diagrams), which are identical except for a ball (resp. a disc) and differs by π rotation of a tangle with 4-ends (resp. tangle diagram with 4-ends) in one of the following ways (see Figure 6.2). The resulting links (resp. diagrams) are called *mutants* of the original link (resp. the original diagram). One can easily see that the following lemma holds.

Lemma 6.2. *A mutant of an adequate diagram is adequate.*

Remark 6.3. We do not know whether a mutant of an adequate link is adequate.

We recall the *Turaev surface* associated to a connected link diagram (see [15], [16] or [37]). First, we construct a cobordism between $s_A D$ and $s_B D$ as follows. Let $\Gamma \subset S^2$ be the underlying 4-valent graph of a connected link diagram D, and V the union of the vertices of Γ. Let $\Gamma \times [-1, 1]$ be a surface with singularities $V \times [-1, 1]$ naturally embedded in $S^2 \times [-1, 1]$. Replace the neighborhoods of $V \times [-1, 1]$ with saddle surfaces positioned in such a way that the boundary curves in $S^2 \times \{1\}$ (resp. $S^2 \times \{-1\}$) correspond to $s_A D$ (resp. $s_B D$) as in Figure 6.3. Figure 6.4 illustrates the construction of the Turaev surface associated to a connected link diagram. The Turaev surface

Figure 6.3: a saddle surface

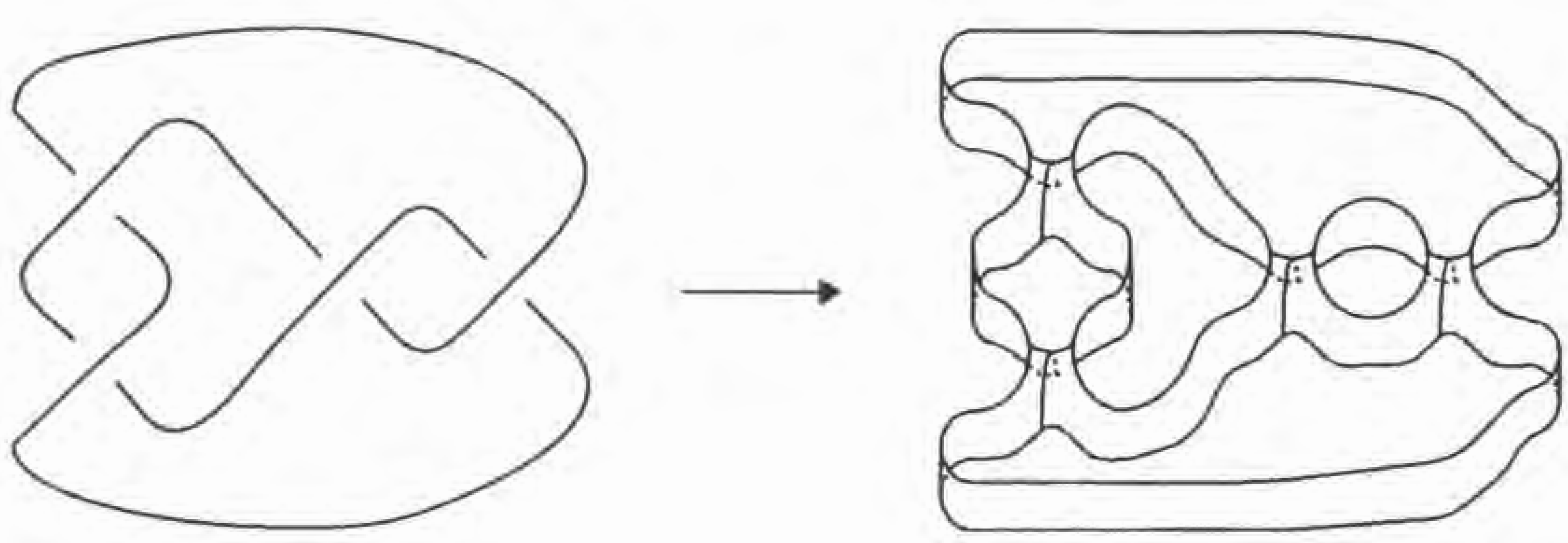

Figure 6.4: on the construction of the Turaev surface of a knot diagram

is completed by attaching disjoint discs to the $|s_A D| + |s_B D|$ boundary circles in S^3.

We denote by $g_T(D)$ the genus of the Turaev surface associated to D. The *Turaev genus* $g_T(L)$ of a non-split link L is the minimal number of the genera $g_T(D)$ of associated to connected diagrams D of the link L. By considering the Euler characteristic of the Turaev surface associated to a connected link diagram, we obtain

Proposition 6.4 ([15], [62]). *Let D be a connected link diagram. Then we have*

$$g_T(D) = \frac{1}{2}(c(D) + 2 - |s_A D| - |s_B D|).$$

The Turaev genus is closely related to algebraic invariants. For a non-split link L, Bae and Morton [8] and Dasbach et al. [16] showed

$$g_T(L) \leq c(L) - \operatorname{span} V_L(t). \tag{6.2}$$

Khovanov [30] introduced an invariant of links, called the Khovanov homology, which values bigraded $\mathbb{Z}$-modules and whose graded Euler characteristic is the Jones polynomial. We denote by $\omega_{Kh}(\mathrm{L})$ the *homological width* (or *thickness*) of the Khovanov homology of a link L. For a knot K, Manturov [39] and Champanerkar, Kofman and Stoltzfus [12] (see also [6]) showed

$$\omega_{Kh}(K) - 2 \leq g_T(K). \tag{6.3}$$

Note that Lowrance found an analogue for the homological width of the Floer homology of a knot [37].

We give a relation between the Turaev guns and the dealternating number.

Theorem 6.5 ([2]). *Let L be a non-split link. Then we have*

$$g_T(L) \leq \mathrm{dalt}(L).$$

Note that a non-split almost alternating link is of Turaev genus one by the above inequality. Therefore $T(3,4)$ and $T(3,5)$ are also of Turaev genus one. Since all non-alternating knots of eleven or fewer crossings except 11_{n95}

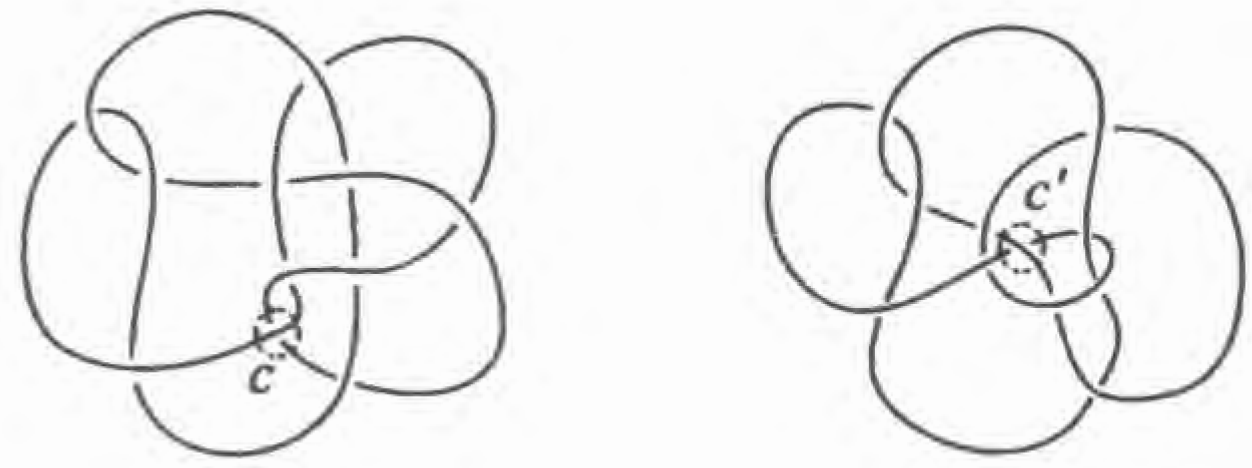

Figure 6.5:

and 11_{n118} are almost alternating, by Theorem 6.5, we have the following.

Corollary 6.6. *Let K be a non-alternating knots of eleven or fewer crossings except 11_{n95} and 11_{n118}. Then we have $g_T(K) = 1$ and $\mathrm{alt}(11_{n95}) = \mathrm{alt}(11_{n118}) = 1$.*

Remark 6.7. In [57], Stošić showed that $\omega_{Kh}(K) \geq 4$ for a non-alternating torus knot K other than $T(3,4)$ and $T(3,5)$. This implies that $g_T(K) \geq 2$ by the inequality (6.3). Therefore a torus knot K is of Turaev genus one if and only if $K = T(3,4)$ or $T(3,5)$.

Theorem 6.8 ([2]). *Let $T_{3,3n+i}$ be the $(3, 3n+i)$-torus knot, where n is a non-negative integer and $i = 1, 2$. Then we have*

$$g_T(T_{3,3n+i}) = \mathrm{dalt}(T_{3,3n+i}) = n.$$

6.3 The Turaev genus of an adequate knot

In this section, we prove Theorem 6.10 and its corollaries.

The following key lemma was implicitly stated and proved by Khovanov [31]. We give a proof of the lemma again in Appendix, which is slightly different from Khovanov's. We use the method for calculating Khovanov homology, which was developed in [57] and [63].

Lemma 6.9 ([31]). *Let K be a knot which admits an adequate diagram D. Then*

$$1/2(c(D) - |s_A D| - |s_B D|) + 3 \leq \omega_{Kh}(K).$$

Here we assume familiarity with Khovanov homology, Lee homology and Lee's spectral sequence. We use the notation of [33] and work over $\mathbb{Q}$.

proof of Lemma 6.9. Suppose D is alternating. Lee [33] showed that $\omega_{Kh}(K) = 2$ and it is well known that $|s_A D| + |s_B D| = c(D) + 2$. Thus the conclusion of the lemma holds for the case where D is alternating. We suppose D is non-alternating. Khovanov [30] showed

$$\overline{\mathcal{H}}^{0,-|s_A D|}(D) \simeq \mathbb{Q} \text{ and } \overline{\mathcal{H}}^{c(D),c(D)+|s_B D|}(D) \simeq \mathbb{Q}.$$

If D is positive, $\overline{\mathcal{H}}^{0,-|s_A D|+2}(D) \simeq \mathbb{Q}$ [31]. By Lee's spectral sequence, we have

$$\dim \overline{\mathcal{H}}^{c(D)-1,c(D)+|s_B D|-4}(D) \geq 1.$$

This means $\omega_{Kh}(K) \geq 1/2(c(D) - |s_A D| - |s_B D|) + 3$. If D is negative, then by taking the mirror image of D, the proof is reduced to the case where D is positive. If D is not positive nor negative, by Lee's spectral sequence, we have

$$\dim \overline{\mathcal{H}}^{1,-|s_A D|+4}(D) \geq 1 \text{ and } \dim \overline{\mathcal{H}}^{c(D)-1,c(D)+|s_B D|-4}(D) \geq 1.$$

This also means $\omega_{Kh}(K) \geq 1/2(c(D) - |s_A D| - |s_B D|) + 3$. □

By Lemma 6.9, we obtain the following.

Theorem 6.10. *Let K be a knot which admits an adequate diagram D. Then*

$$g_T(K) = g_T(D) = 1/2(c(D) - |s_A D| - |s_B D|) + 1 = \omega_{Kh}(K) - 2 = c(K) - \operatorname{span} V_K(t).$$

Proof. We have the following chain of inequalities:

$$\begin{aligned} g_T(K) &\leq g_T(D) \\ &= 1/2(c(D) - |s_A D| - |s_B D|) + 1 \\ &\leq \omega_{Kh}(K) - 2 \\ &\leq g_T(K). \end{aligned}$$

Here these inequalities follow from the definition of the Turaev genus of a knot, Proposition 6.4, Lemma 6.9 and the inequality (6.3). Therefore all the above inequalities are in fact equalities. The remaining equality in the statement of this theorem, namely that regarding $c(K) - \operatorname{span} V_K(t)$, follows at once from Proposition 6.1. □

The following is a direct corollary of Theorem 6.10.

Corollary 6.11. *Let K be a knot which satisfies one of the following inequalities*

(i)

$$g_T(K) < c(K) - \operatorname{span} V_K(t), \tag{6.4}$$

(ii)

$$\omega_{Kh}(K) - 2 < g_T(K). \tag{6.5}$$

Then K is not adequate.

Remark 6.12. The inequality (6.4) is relatively effective. For example, the inequality (6.4) holds all non-adequate knots up to 11 crossings. Manolescu and Ozsváth [38] introduced a class of *quasi-alternating knots*, which includes alternating knots. For the definition of a quasi-alternating knot, see [38]. Let K be a quasi-alternating knot. Then $\omega_{Kh}(K) = 2$ [38]. If K is not alternating, then K is not adequate by inequality (6.5). Equivalently, non-alternating adequate knot is not quasi-alternating.

We obtain the additivity of the Turaev genus of adequate knots.

Corollary 6.13. *Let K and K' be adequate knots. Then*

$$g_T(K\#K') = g_T(K) + g_T(K').$$

Proof. Let D and D' be adequate diagrams of K and K' respectively. We have $c(D\#D') = c(D) + c(D')$ and it is easy to see that $|s_A(D\#D')| = |s_AD| + |s_AD'| - 1$ and $|s_B(D\#D')| = |s_BD| + |s_BD'| - 1$. Note that $D\#D'$ is an adequate diagram. By Proposition 6.4 and Theorem 6.10, $g_T(K\#K')$ is equal to

$$\begin{aligned}
g_T(D\#D') &= \frac{1}{2}(c(D\#D') + 2 - |s_A(D\#D')| - |s_B(D\#D')|) \\
&= \frac{1}{2}(c(D) + c(D') + 2 - |s_AD| - |s_AD'| + 1 - |s_BD| - |s_BD'| + 1) \\
&= g_T(D) + g_T(D') \\
&= g_T(K) + g_T(K').
\end{aligned}$$

□

The following corollary suggests that the Turaev genus of a knot might be preserved under mutation.

Corollary 6.14. *Let K be a knot admitting an adequate diagram D and K' a knot which has a mutant of D. Then*

$$g_T(K) = g_T(K').$$

Proof. Let D' be the mutant of D. We have $c(D) = c(D')$ and one can easily see that $|s_AD| = |s_AD'|$ and $|s_BD| = |s_BD'|$ by the definition of the mutation. By Proposition 6.4 and Theorem 6.10, we obtain

$$\begin{aligned}
g_T(K) &= g_T(D) \\
&= \frac{1}{2}(c(D) + 2 - |s_AD| - |s_BD|) \\
&= \frac{1}{2}(c(D') + 2 - |s_AD'| - |s_BD'|) \\
&= g_T(D').
\end{aligned}$$

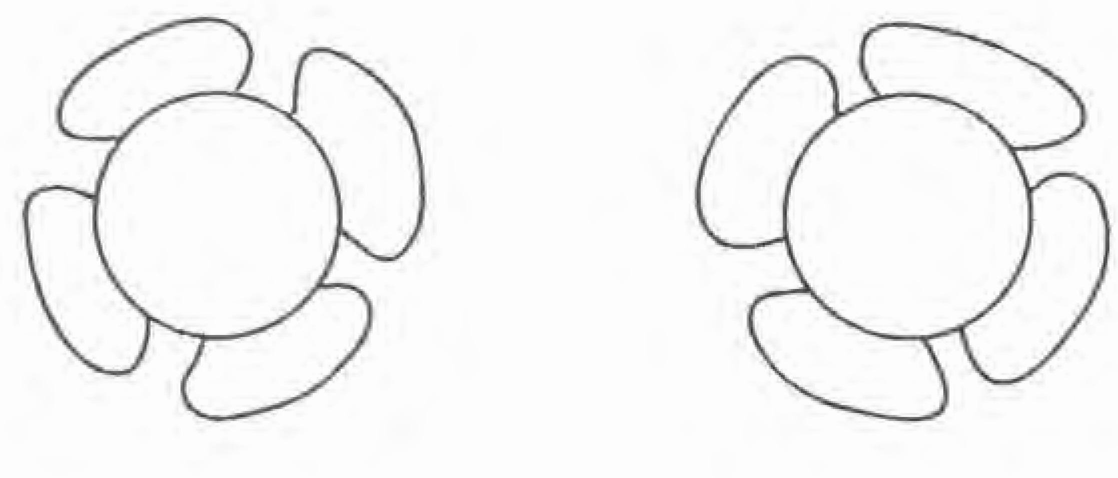

Figure 6.6: the two planar closures for a 4-tangle diagram

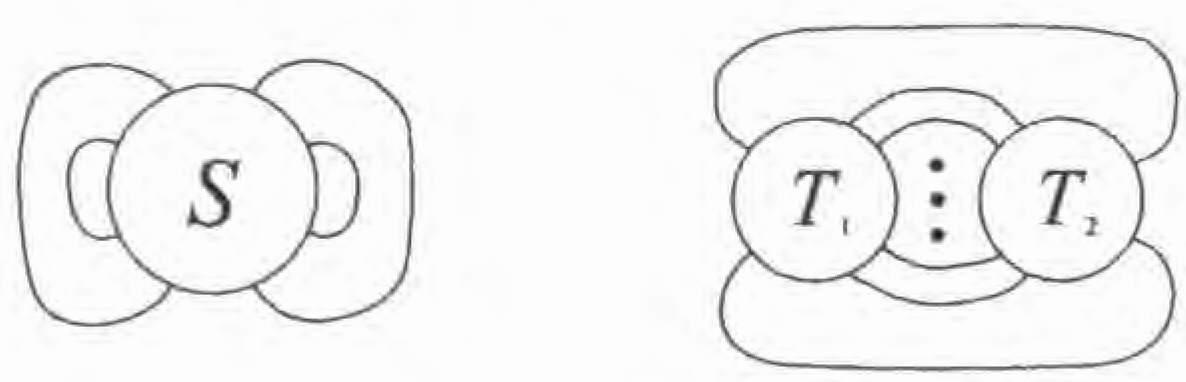

Figure 6.7: a planar closure of a 4-tangle diagram S and n-semi-alternating diagram

By Lemma 6.2, D' is an adequate diagram. Thus we have $g_T(D') = g_T(K')$. □

We recall the definition of an *n-semi-alternating link* [9] and prove Theorem 6.19. An *n-tangle diagram* is a tangle diagram with $2n$ ends. An n-tangle diagram is *strongly alternating* if each of two planar closures of the tangle diagram which connects adjacent two ends of the tangle diagram (see Figure 6.6). An *n-semi-alternating diagram D* is a non-alternating diagram which is a "sum" of two strongly alternating $(n+1)$-tangle diagrams by gluing $2n+2$ ends as in Figure 6.7, where n is a positive integer. An *n-semi-alternating link* [9] is a link which has an n-semi-alternating diagram. For a tangle diagram, there are many possible planar closures [1] of the tangle diagram in general. The diagram in Figure 6.7 is one of possible closures of a 4-tangle diagram.

Lemma 6.15. *If an n-tangle diagram is strongly alternating, then any planar closure of the tangle diagram is connected reduced alternating.*

[1]The number of possible closures of the tangle diagram is the n-th Catalan number, which is equal to $\binom{2n}{n}/(n+1)$.

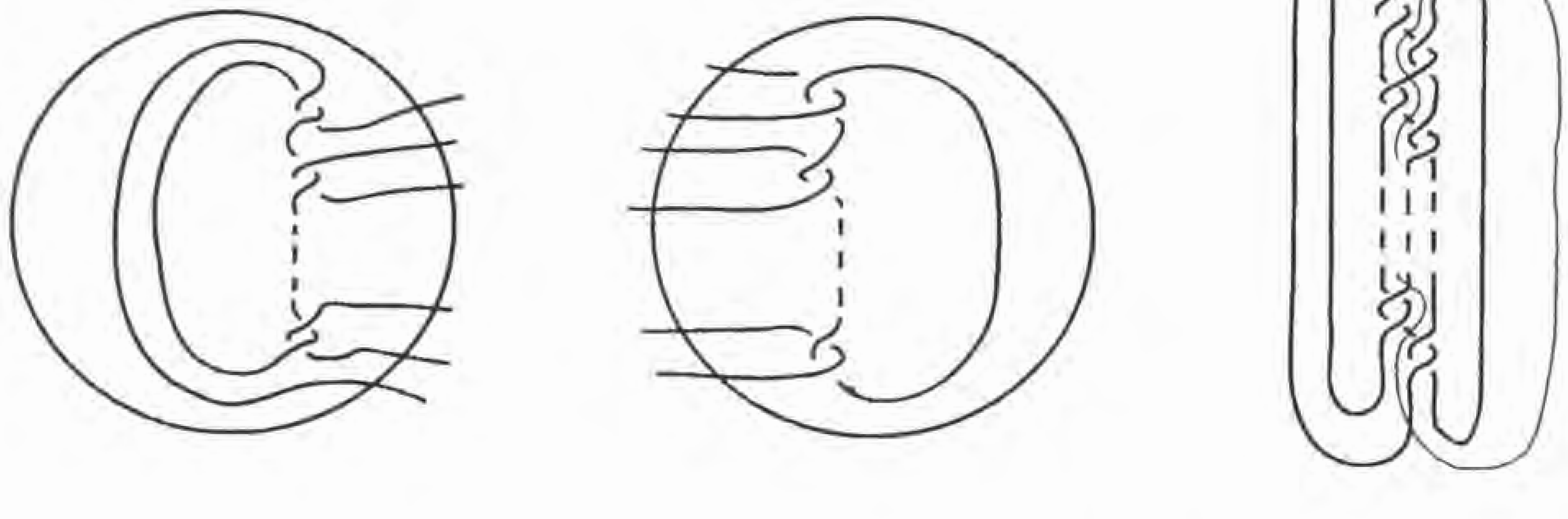

Figure 6.8: an example of n-semi-alternating diagram

Proof. If there exists a planar closure X of the tangle diagram which is not connected, then there exists a loop l in $S^2 \setminus X$ separating S^2 each containing part of X. One of the two planar closures of the tangle diagram which connects adjacent two ends of the tangle diagram has no intersection with l. This is a contradiction. Thus every planar closure of the tangle diagram is connected. We can prove that every planar closure of the tangle diagram is reduced in the same way. □

A 1-semi-alternating link is called *semi-alternating* in [35]. One of the typical semi-alternating links is a pretzel link $P(p_1, \cdots p_n, q_1, \cdots, q_m)$, where $n, m \geq 2$ and $p_i, q_j \geq 2$ for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. The following is a typical n-semi-alternating link.

Example 6.16. *Let β be a 3-braid of the form $\prod_{i=1}^{n+1} \sigma_1^{p_i} \sigma_2^{q_i}$ such that $p_i, q_i \geq 2$ for $i = 1, 2, \ldots, n+1$. Then the closure of β is an n-semi-alternating link (see Figure 6.8).*

A semi-alternating diagram is adequate [35]. We show that an n-semi-alternating diagram is adequate since no proof was given in [9]. The proof is essentially the same as that of Proposition 4 in [35].

Proposition 6.17. *An n-semi-alternating diagram is adequate.*

Proof. Let D be an n-semi-alternating diagram. If D is not adequate, then there exist two segments near a crossing c in D which lie in the same component of $s_A(D)$ or $s_B(D)$. Since the crossing c is in one of the original two

Figure 6.9: a checkerboard coloring of D

tangles, the appropriate planar closure X of the tangle which contains c can be chosen so that the two segments lie in the same component of $s_A(X)$ or $s_B(X)$. This means that X is not adequate. On the other hand, X is a reduced alternating diagram by Lemma 6.15. Thus X is adequate. This is a contradiction. □

The following proposition was proved for the case $n = 1$ in [35]. We also generalize this result to an n-semi-alternating link.

Proposition 6.18. *Let L be an n-semi-alternating link. Then*

$$\operatorname{span} V_L(t) = c(L) - n.$$

Proof. Let D be an n-semi-alternating diagram of L which is composed of two strongly alternating tangle diagrams T_1 and T_2. Let the regions of D be colored black and white in checkerboard fashion. Without loss of generality, we assume that the crossings of T_1 and T_2 are as in Figure 6.9. Let D_1 and D_2 be diagrams which are the closures of tangle diagrams T_1 and T_2 as in Figure 6.10. For $i = 1, 2$, we denote by b_i the numbers of black regions in D_i and by w_i the numbers of white regions in D_i. In $s_A D$ there is a single simple closed curve that contains all $2n+2$ of the arcs connecting T_1 and T_2. Each other component of $s_A D$ enclosed one of the $b_1 - (n+1)$ black regions of T_1 not incident upon these connecting arcs or one of the corresponding $w_2 - 1$ white regions of T_2. Thus $|s_A D| = (b_1 - (n+1)) + (w_2 - 1) + 1 = b_1 + w_2 - (n+1)$. Similarly we obtain $|s_B D| = w_1 + b_2 - (n+1)$. We have

$$|s_A D| + |s_B D| = w_1 + w_2 + b_1 + b_2 - 2(n+1).$$

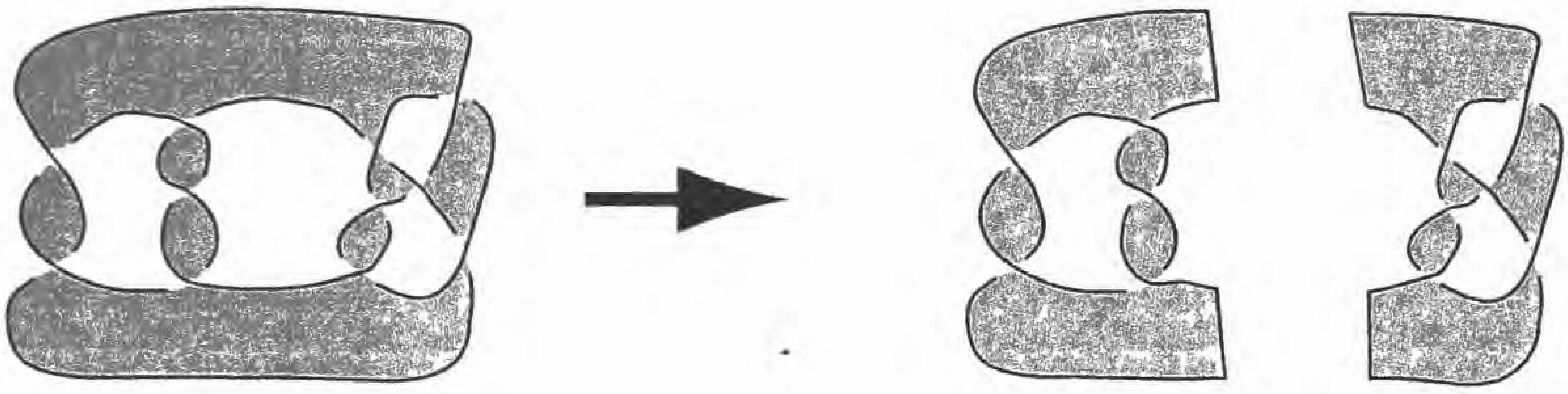

Figure 6.10: the definition of D_1 and D_2.

Since D_i is connected alternating, we have $w_i + b_i = c(D_i) + 2$ for $i = 1, 2$. Thus

$$\begin{aligned} &= c(D_1) + c(D_2) - 2n + 2 \\ &= c(D) - 2n + 2. \end{aligned}$$

By Proposition 6.1, the span of the Jones polynomial of L is equal to

$$\frac{1}{2}(c(D) + |s_A D| + |s_B D| - 2) = c(L) - n.$$

□

Finally, we prove the following.

Theorem 6.19. *Let K be an n-semi-alternating knot. Then*

$$g_T(K) = \mathrm{dalt}(K) = n.$$

Proof. By Theorem 6.10 and Proposition 6.18, we have $g_T(K) = c(K) - \operatorname{span} V_K(t) = n$. Let D be an n-semi-alternating diagram of K. By deforming D as in Figure 6.11, one can easily see that $\mathrm{dalt}(K) \leq n$. By Theorem 6.5c, we obtain $n = g_T(K) \leq \mathrm{dalt}(K) \leq n$. □

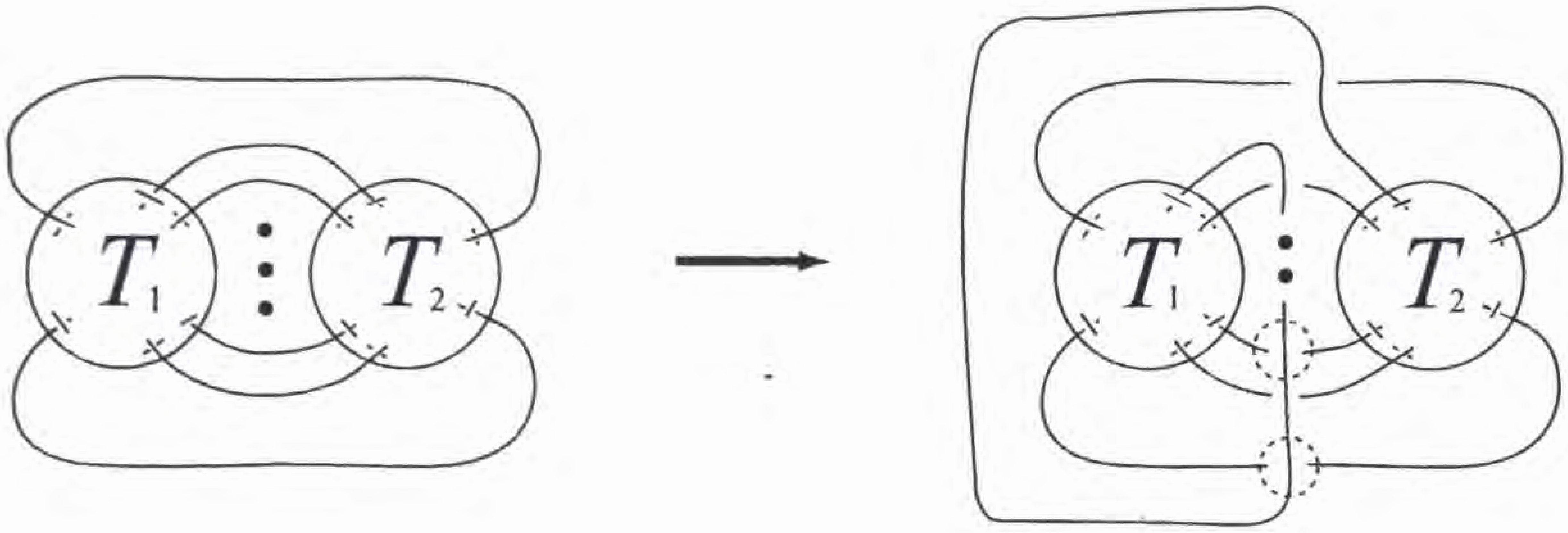

Figure 6.11: a deformation of an n-semi-alternating diagram

6.4 Questions on the Turaev genus of a knot

Question 1. *For any knot K, is it true that $g_T(K) = \mathrm{dalt}(K)$?*

This equality holds for alternating knots, almost alternating knots which includes non-alternating Montesinos knots and semi-alternating knots, knots up to 11 crossings except 11_{n95} and 11_{n118}, the $(3, q)$-torus knots [2] and n-semi-alternating knots (Theorem 6.19).

Question 2. *Is the Turaev genus of a knot invariant under mutation?*

The Turaev genus of a knot is invariant under mutation for alternating knots, Montesinos knots and torus knots since a mutant of an alternating knot is an alternating knot [56], a mutant of a Montesinos knot is a Montesinos knot [56] and a torus knot is unchanged by mutation for the fundamental group considerations [49].

Let $KT_{r,n}$ be a Kinoshita-Terasaka knot (see [51]), indexed by integer $|r| > 1$ and $n > 0$ and $C_{r,n}$ be the Conway knot, which is obtained from $KT_{r,n}$ by mutation. As suggested in [35], these knots are semi-alternating. Thus we have $g_T(KT_{r,n}) = g_T(C_{r,n}) = 1$.

Question 3. *Is the Turaev genus of a knot additive under connected sum, that is, does the equality $g_T(K\#K') = g_T(K) + g_T(K')$ hold?*

A partial result is Corollary 6.13.

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